ON THE SET OF GAMES ON THE UNIT SQUARE WITH A UNIQUE SOLUTION

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Abstract

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ON THE SET OF GAMES ON THE UNIT SQUARE WITH A UNIQUE SOLUTION

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Let C be the standard metrized space of continuous functions on the square $X \times Y$, where $X, Y = [0, 1]$. Each function $K \in C$ determines a two-person zero-sum game Γ_K on the unit square, in which $K(x, y)$ is the payoff of the first player if he has chosen x, and the second player has chosen y. As Ville showed (^{4}) (see also (^{1})), every such game has an equilibrium situation in pure strategies. We shall say that a game has a unique solution if each player has only one optimal strategy. A game with a unique solution will be called a game with finite spectrum if the spectrum of the players' optimal strategies is finite. All necessary definitions may be found in (^{1}). As was shown in (^{1,2}), the set of matrices of a given size that determine a game with a unique solution is an open and dense subset of the set of all matrices of the given size.

Theorem 1. The set of functions $K \in C$ that determine a game with finite spectrum is a dense subset of C.

Theorem 2. The set of functions $K \in C$ for which the game Γ_K has the following properties:

1) Γ_K has a unique solution;

2) the optimal distribution functions are continuous;

3) the spectrum of the optimal strategy of each player is a nowhere dense perfect closed set of Lebesgue measure zero, containing an everywhere dense subset of type G_δ.

Here we shall give a proof of a weaker theorem.

Theorem 2′. The set of functions $K \in C$ that determine a game with a unique solution is everywhere dense of type G_δ.

It is easy to construct an example showing that the set of functions $K \in C$ determining a game with a unique solution is not open.

Proof of Theorem 1. Denote by $\{K\}_n$ the class of all continuous functions that are linear on the triangles with vertices $(i/n, j/n)$, $((i+1)/n, j/n)$, $((i+1)/n, (j+1)/n)$, and on the triangles with vertices $(i/n, j/n)$, $(i/n, (j+1)/n)$, $((i+1)/n, (j+1)/n)$, $i, j = 0, 1, \dots, n-1$. Next put

$$A_n = \{x \in X \mid x = i/n,\ i = 0, 1, \dots, n\},$$

$$B_n = \{y \in Y \mid y = j/n,\ j = 0, 1, \dots, n\}.$$

Then, for any $\varepsilon > 0$, one can find a number n and a function $K_n \in \{K\}_n$ such that the following conditions are satisfied:

1) the game Γ_n, specified on $A_n \times B_n$ by the payoff function $K_n(x, y)$, has a unique solution.
2) $\rho(K_n, K) < \varepsilon$.

The existence of such a function follows from the uniform continuity of the function K, and also from the fact that the set of matrix games with a unique solution is everywhere dense (see, for example, (^{1})). Let F_n and G_n be optimal –

strategies of the players in the game Γ_n. Since $A_n \subset X$, and $B_n \subset Y$, the strategies F_n and G_n determine in a natural way measures on X and Y. We shall denote them also by F_n and G_n. These strategies are optimal in the game Γ_{K_n}. Let us verify this. Indeed, the minimum of the function

$$\varphi(y) = \int K_n(x, y)\, dF_n(x)$$

is attained on B_n, since on each interval complementary to B_n the function $\varphi(y)$ is linear, and therefore the value of the minimum is not less than the value of the game Γ_n, because F_n is an optimal strategy in the game Γ_n. Thus, by using the strategy F_n, the first player can guarantee himself a payoff not less than V_{Γ_n}, where V_{Γ_n} is the value of the game Γ_n. It is shown analogously that, by using the strategy G_n, the second player cannot lose more than V_{Γ_n}. This means that the strategies F_n and G_n are optimal strategies in the game Γ_{K_n}.

Let $\Phi_n(x, y)$ be a continuous function on $X \times Y$, equal to zero on $A_n \times B_n$. Suppose, moreover, that $\Phi_n(x, y) > 0$ on $A_n \times Y \setminus A_n \times B_n$ and $\Phi_n(x, y) < 0$ on $X \times B_n \setminus A_n \times B_n$. Obviously, such a function exists. The value of the game Γ_{Φ_n} is equal to zero, and the optimal strategies of the players are derivative measures respectively on A_n and B_n, including F_n and G_n. Consequently, F_n and G_n are optimal strategies in the game Γ_P, where $P_n = K_n + \alpha_n \Phi_n$, α_n is an arbitrary positive number, and the value of the game is equal to V_{Γ_n}. These strategies are the unique optimal strategies in the game Γ_{P_n}. Indeed,

$$\int P_n(x, y)\, dF_n(x) > V_{\Gamma_n}$$

on $Y \setminus B_n$. Therefore the support of any optimal strategy of the second player is contained in B_n. It is shown analogously that the support of any optimal strategy of the first player is contained in A_n. Thus, all optimal strategies of the players in the game Γ_{P_n} are optimal strategies in the game Γ_n. The game Γ_n has a unique solution; hence the game Γ_{P_n} has a unique solution. Since ε and α_n are arbitrary positive numbers, the theorem is proved.

Proof of Theorem $2'$**.** Let $\hat{\rho}(F_1, F_2)$ be some metric on the set of distribution functions, convergence in which is equivalent to convergence in the main. For example, as $\hat{\rho}$ one may take the Lévy metric (see (3)). It can also be defined as

$$\max_x \left| \int_0^x F_1(t)\,dt - \int_0^x F_2(t)\,dt \right|.$$

We shall say that the game Γ_K has an ε-unique solution if the set of optimal strategies of the first player can be enclosed in an open ball of radius ε, and the set of optimal strategies of the second player can be enclosed in an open ball of radius ε. Let M_ε be the set of continuous functions K for which the game Γ_K has an ε-unique solution. We shall prove that M_ε is an open subset of C. Indeed, let $K \in M_\varepsilon$, $K_m \in C \setminus M_\varepsilon$, and suppose that the sequence K_m converges to K. Moreover, let A_j, $j = 1, 2$, be an open ball of radius ε which contains all optimal strategies of player j in the game Γ_K. We may assume that there exists a player j such that, for every m, in the game Γ_{K_m} he has an optimal strategy F_m not contained in A_j; otherwise one could choose a subsequence of the sequence K_m possessing this property. From the sequence F_m one can choose a subsequence converging in the main, and hence also in the metric $\hat{\rho}$, to some optimal strategy of player j in the game Γ_K, contained in A_j, which is impossible, since $F_m \notin A_j$, and A_j is an open set. The contradiction obtained proves the openness of M_ε. The set of all functions $K \in C$ for which the game Γ_K has a unique solution is equal to $\bigcap M_{1/n}$. Each set $M_{1/n}$ is open and contains all K that define a game with a unique solution,

and therefore, by Theorem 1, is dense in C. Thus, the set of all K for which the game Γ_K has a unique solution is the intersection of a countable number of open sets that are everywhere dense. Theorem $2'$ is proved.

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Note: Figure translations are in progress. See original paper for figures.