FOURIER SERIES WITH RESPECT TO FUNDAMENTAL SYSTEMS OF FUNCTIONS OF THE POLYHARMONIC OPERATOR

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Abstract

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FOURIER SERIES WITH RESPECT TO FUNDAMENTAL SYSTEMS OF FUNCTIONS OF THE POLYHARMONIC OPERATOR

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In the present paper we study questions of localization and convergence of Fourier series with respect to the so-called fundamental systems of functions of the polyharmonic operator Δ^m.

The concept of a fundamental system of functions of the polyharmonic operator includes, as a special case, the system of eigenfunctions of any nonnegative self-adjoint extension in L_2 of the minimal operator generated by the operator $(-1)^m\Delta^m$. For the indicated Fourier series we establish: 1) for an arbitrary N-dimensional domain, definitive localization conditions in the Sobolev classes W_2^α; 2) for an arbitrary odd-dimensional domain, definitive localization and convergence conditions in the classes $C^{(n,\alpha)}$; 3) for an arbitrary even-dimensional domain, localization and convergence conditions in the classes $C^{(n,\alpha)}$ that are close to definitive.

To prove all the results obtained in this paper, we use the method developed in paper ([1]), in conjunction with the mean-value theorems for a regular solution of the equation $\Delta^m u - (-1)^m \lambda u = 0$, established in paper ([2]).

We proceed to a precise formulation of the results. Let G be an arbitrary N-dimensional domain, and let Ω be any subdomain of it*.

A complete orthonormal system $\{u_k(x)\}$ in the domain G will be called a **fundamental system of functions** (f.s.f.) of the **polyharmonic operator in the subdomain** Ω, if each function $u_k(x)$ belongs inside Ω to the class $C^{(2m)}$ and, for some nonnegative number λ_k, satisfies inside Ω the equation $\Delta^m u_k - (-1)^m \lambda_k u_k = 0$.

The numbers λ_k will be called **fundamental numbers**, and the corresponding functions $u_k(x)$, **fundamental functions**.

The study of f.s.f. frees one from prescribing boundary conditions in any form and makes it possible not to make any assumptions either about the smoothness and differential properties of the fundamental functions outside the domain Ω, or about the smoothness of the boundary Γ of the domain G.

An f.s.f. of the polyharmonic operator in the subdomain Ω is the system of eigenfunctions of some self-adjoint nonnegative extension, with range in the wider space $L_2(G)$, of the minimal operator generated by the operator $(-1)^m\Delta^m$.

In what follows, for simplicity, we shall assume that the fundamental numbers have no finite points of condensation, and shall regard the fundamental numbers as numbered in increasing order.

When studying questions of localization and convergence of Fourier series, we shall consider functions $f(x)$ that are finite in the domain Ω (i.e., have support compact in the domain Ω). This circumstance will be indicated below, as usual, by the symbol $\circ$ in the notation of the classes.

* Here and below the subdomain Ω may coincide with the whole domain G.

For every such function $f(x)$, the divergence of its Fourier series at a fixed point x_0 of the domain Ω may be due to two causes: 1) insufficient smoothness of $f(x)$ in a small neighborhood of the point x_0 (i.e., the absence of local smoothness); 2) insufficient smoothness of $f(x)$ in the whole domain Ω (i.e., the absence of smoothness ensuring localization of the Fourier series).

First assume the function $f(x)$ to be arbitrarily smooth (equal to zero) in a small neighborhood of a fixed point x_0, and establish those (as far as possible maximal) smoothness requirements on $f(x)$ everywhere in Ω which still do not ensure localization of the Fourier series (i.e., convergence of this series at the point x_0).

Theorem 1 (on conditions not ensuring localization in the classes $C^{(n,\alpha)}$). *Let $N \geq 2$, let G be an arbitrary N-dimensional domain, let $\{u_k(x)\}$ be an arbitrary f.s.f. of the polyharmonic operator in some subdomain Ω of it; let x_0 be any interior point of Ω; and let α be any fixed number satisfying the inequalities*

$$\alpha < 1 \quad \text{for odd } N,$$

$$\alpha < 1/2 \quad \text{for even } N. \tag{1}$$

Then there exists a function $f(x)$ satisfying the following conditions: 1) $f(x)$ vanishes in some neighborhood D of the point x_0; 2) $f(x) \in C_0^{([N/2]-1,\alpha)}(\Omega)$; 3) the Fourier series of the function $f(x)$ with respect to the system $\{u_k(x)\}$ diverges at the point x_0.[1]

Corollary (on conditions not ensuring localization in the classes W_2^α). *Let $N, G, \Omega, \{u_k(x)\}$, and x_0 have the same meaning as in Theorem 1; let α be any fixed real number satisfying the inequality $\alpha < (N-1)/2$.*

[1]When summing in the order of increasing fundamental numbers.

Then there exists a function $f(x)$ satisfying the following conditions: 1) $f(x)$ vanishes in some neighborhood D of the point x_0; 2) $f(x) \in \mathring{W}_2^{\alpha}(\Omega)$; 3) the Fourier series of the function $f(x)$ with respect to the system $\{u_k(x)\}$ diverges at the point x_0.

Thus, membership of the function $f(x)$ in the class $C_0^{([N/2]-1,\alpha)}(\Omega)$ for α satisfying the inequalities (1), or membership of $f(x)$ in the class $\mathring{W}_2^{\alpha}(\Omega)$ for $\alpha < (N-1)/2$, ensures neither convergence nor even localization of the Fourier series.

Theorem 2 (on conditions ensuring convergence of the Fourier series). *Let $N \geq 2$; let G be an arbitrary N-dimensional domain; let $\{u_k(x)\}$ be an arbitrary f.s.f. of the polyharmonic operator in some subdomain Ω of it; and let $f(x)$ be an arbitrary function satisfying the following two requirements: 1) $f(x) \in \mathring{W}_2^{(N-1)/2}(\Omega)$; 2) in some domain D contained in Ω, the function $f(x)$ belongs to the class $W_p^{(N-1)/2}$ for $p > 2N/(N-1)$ when N is odd and to the class $W_{2N/(N-1)}^{\alpha}$ for $\alpha > (N-1)/2$ when N is even.*

Then the Fourier series of the function $f(x)$ converges to this function everywhere inside D, and its convergence is uniform in every strictly interior subdomain D' of the domain D.

Corollary 1 (on conditions ensuring localization in the classes W_2^{α}). *Let N, G, Ω, and $\{u_k(x)\}$ have the same meaning as in Theorem 2; let $f(x)$ be an arbitrary function satisfying two requirements: 1) $f(x)$ vanishes in some domain D contained in Ω; 2) $f(x) \in \mathring{W}_2^{(N-1)/2}(\Omega)$.*

Then the Fourier series of the function $f(x)$ with respect to the system $\{u_k(x)\}$ converges to zero inside D, and its convergence is uniform in every strictly interior subdomain D' of the domain D.

Comparison of the corollary just formulated with the corollary from

Theorem 1 shows that for an arbitrary N-dimensional domain we have established final localization conditions in the Sobolev classes W_2^{α} for a finite function $f(x)$: for $\alpha \geq (N-1)/2$ the localization principle is valid, while for $\alpha < (N-1)/2$ localization does not occur.

Corollary 2 (on conditions ensuring convergence in the classes $C^{(n,\alpha)}$). Let N, G, Ω, and $\{u_k(x)\}$ have the same meaning as in Theorem 2; let $f(x)$ be an arbitrary function satisfying the requirement: $f(x)$ belongs to the class $C_0^{([N/2]-1,\alpha)}(\Omega)$ for

$$\alpha = 1 \text{ for odd } N,$$

$$\alpha > \tfrac{1}{2} \text{ for even } N.$$

Then the Fourier series of the function $f(x)$ with respect to the system $\{u_k(x)\}$ converges to $f(x)$ everywhere inside Ω, and its convergence is uniform in every strictly interior subdomain Ω' of the domain Ω.

Comparison of this corollary with Theorem 1 shows that for an arbitrary odd-dimensional domain we have established final localization and convergence conditions in the classes $C^{(n,\alpha)}$ for the Fourier series of a finite function $f(x)$: membership of $f(x)$ in the class $C_0^{((N-3)/2,\alpha)}$ for $\alpha < 1$ is insufficient even for localization of the Fourier series, while membership of $f(x)$ in the same class for $\alpha = 1$ is sufficient not only for localization but also for uniform convergence of the Fourier series.

For even N, in the same classes one obtains localization and convergence conditions close to final ones: membership of $f(x)$ in the class $C_0^{((N-2)/2,\alpha)}$ for $\alpha < \frac{1}{2}$ does not ensure even localization, while for $\alpha > \frac{1}{2}$ it ensures not only localization but also uniform convergence of the Fourier series.

Comparing the obtained results with the main results of (1), we arrive at the conclusion that the smoothness requirements on the function $f(x)$ that ensure localization and convergence of its Fourier series do not depend on the order of the elliptic operator under consideration, but depend only on the dimension of the domain.

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Note: Figure translations are in progress. See original paper for figures.