ON A VARIATIONAL METHOD FOR THE BOUNDARY-VALUE PROBLEM OF WAVE PROPAGATION

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ON A VARIATIONAL METHOD FOR THE BOUNDARY-VALUE PROBLEM OF WAVE PROPAGATION

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Consider the problem of wave propagation in a certain three-dimensional region W, containing a rectilinear z-axis and bounded by some surface Π. We shall assume that all possible sections $S(z_i)$ of the interior of the waveguide W by planes $z = z_i$ perpendicular to the z-axis are simply connected and bounded.

The propagation of a monochromatic wave in the region W is described by the Helmholtz equation

$$\Delta u(\mathbf{r}) + k^2 u(\mathbf{r}) = 0, \qquad \mathbf{r} \in W, \tag{1}$$

in which the wave number k will be assumed independent of $\mathbf{r}$. In the problem of an acoustic wave the function u is the velocity potential $\mathbf{V}$ ($\mathbf{V} = \operatorname{grad} u$), and the boundary condition corresponding to an ideally rigid boundary should be taken in the form $\partial u/\partial n|_\Pi = 0$ ($\mathbf{n}$ is the outward normal to Π). In the more general case, equation (1) should be considered with the boundary condition

$$[\partial u/\partial n + \alpha(\Pi)u]_\Pi = 0. \tag{2_1}$$

A boundary condition of the form

$$u|_\Pi = 0, \tag{2_2}$$

corresponding to an ideally soft surface Π, is also possible.

Consider the region W', bounded by the sections $S(z_0)$, $S(z_1)$ for arbitrary values z_0 and z_1 ($z_0 < z_1$) and by the piece Π' of the surface Π lying between the planes $z = z_0$ and $z = z_1$. By direct computation of the variation of the functional

$$I(u) = \iiint_{W'} \left[(\operatorname{grad} u)^2 - k^2 u^2\right] \, dw + \iint_{\Pi'} \alpha u^2 \, d\sigma \tag{3}$$

on the set of functions taking prescribed values in $S(z_0)$ and $S(z_1)$, it is easy to verify that the solutions of problem (1)–(2_1) with additional conditions in the indicated sections are extrema of (3). In this case condition (2_1) turns out to be natural. When solving problem (1)–(2_2), however, in expression (3) one should put $\alpha = 0$ and take care that condition (2_2), which is not natural, is fulfilled.

Assuming that the surface Π is smooth and can be defined by the equation $\rho = R(z, \varphi)$ (cylindrical coordinates), we shall seek the solution of the problem, using the idea of work ([1]), in the form

$$u(\mathbf{r}) = \sum_{m,n} \eta_m^n(\mathbf{r}) f_m^n(z), \tag{4}$$

where $\{\eta_m^n(\mathbf{r})\}$ is some prescribed system of functions, on which in problem (1) –(2_2) the additional requirements $\eta_m^n(\Pi) \equiv 0$ are imposed.

After substituting (4) into (3) and subsequently varying under the above-indicated conditions in $S(z_0)$ and $S(z_1)$, we arrive at a boundary-value problem for a system of ordinary differential equations

$$\frac{d}{dz}\left[\sum_{m',n'} G_{mm'}^{nn'} \frac{df_{m'}^{n'}}{dz} + \sum_{m',n'} F_{mm'}^{nn'} f_{m'}^{n'}\right] - \sum_{m',n'} F_{m'm}^{n'n} \frac{df_{m'}^{n'}}{dz} + \sum_{m',n'} \Phi_{mm'}^{nn'} f_{m'}^{n'} = 0. \tag{5}$$

Here

$$G_{mm'}^{nn'}(z) = \iint_{S(z)} \left(\eta_m^n \eta_{m'}^{n'}\right) \, dS;$$

$$\Phi_{mm'}^{nn'}(z) = k^2 G_{mm'}^{nn'} - \iint_{S(z)} \left(\operatorname{grad} \eta_m^n, \operatorname{grad} \eta_{m'}^{n'}\right) \, dS - \int_0^{2\pi} \left(\eta_m^n \eta_{m'}^{n'}\right)\Big|_\Pi \frac{aR \, d\varphi}{C_z C_\varphi};$$

$$F_{mm'}^{nn'} = \iint_{S(z)} \left(\eta_m^n \frac{\partial \eta_{m'}^{n'}}{dz}\right) \, dS;$$

$$C_z = |\cos(\tau_z, e_z)|, \qquad C_\varphi = |\cos(\tau_\varphi, e_\varphi)|;$$

τ_z and τ_φ are unit vectors tangent to Π, with τ_z lying in the plane passing through the z-axis, and τ_φ in the plane perpendicular to this axis.

Retaining in (4) only a finite number of terms ($m = 0, 1, 2, \dots, M-1;\ n = 0, 1, 2, \dots, N-1$), we arrive at system (5) with a finite number of equations. In this case the finite systems of functions $G_{mm'}^{nn'}$, $\Phi_{mm'}^{nn'}$, and $F_{mm'}^{nn'}$ can be arranged in the form of square matrices G, Φ, and F, and the unknowns f_m^n in the form of a vector $\mathbf{f}$, after which system (5) can be written in the form

$$\frac{d}{dz}\left[G\frac{d\mathbf{f}}{dz} + F\mathbf{f}\right] - F^{\tau}\frac{d\mathbf{f}}{dz} + \Phi\mathbf{f} = 0, \qquad (6)$$

where the superscript τ denotes transposition. The arrangement of the elements in the indicated matrices may be, for example, as follows:

$$A = \|A_{mm'}\| = \begin{pmatrix} A_{00}A_{01} & \dots & A_{0\,M-1} \\ A_{10}A_{11} & \dots & A_{1\,M-1} \\ \dots & \dots & \dots \\ A_{M-1\,0}A_{M-1\,1} & \dots & A_{M-1\,M-1} \end{pmatrix},$$

$$A_{mm'} = \|A_{mm'}^{nn'}\| = \begin{pmatrix} A_{mm'}^{00} & A_{mm'}^{01} & \dots & A_{mm'}^{0\,N-1} \\ A_{mm'}^{10} & A_{mm'}^{11} & \dots & A_{mm'}^{1\,N-1} \\ \dots & \dots & \dots & \dots \\ A_{mm'}^{N-1\,0} & A_{mm'}^{N-1\,1} & \dots & A_{mm'}^{N-1\,N-1} \end{pmatrix};$$

$$\mathbf{f} = (f_0^0, f_0^1, \dots, f_0^{N-1}, f_1^0, f_1^1, \dots, f_1^{N-1}, \dots, f_{M-1}^0, f_{M-1}^1, \dots, f_{M-1}^{N-1})^{\tau}.$$

In view of the fact that $G_{mm'}^{nn'} = G_{m'm}^{n'n}$ and $\Phi_{mm'}^{nn'} = \Phi_{m'm}^{n'n}$, the matrices G and Φ are symmetric: $G = G^{\tau}$ and $\Phi = \Phi^{\tau}$.

Setting $\mathbf{g} = G\dfrac{d\mathbf{f}}{dz} + F\mathbf{f}$, $\mathbf{h} = (f_0^0, \dots, f_{M-1}^{N-1}, g_0^0, \dots, g_{M-1}^{N-1})^{\tau}$, we arrive at the Hamiltonian form of system (6):

$$J\,d\mathbf{h}/dz = H(z)\mathbf{h}, \qquad (7)$$

where $J = -J^{\tau}$ and $H(z) = H(z)^{\tau}$ are real square matrices of order $2MN$:

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \qquad H(z) = \begin{pmatrix} \Phi + F^{\tau}G^{-1}F & -F^{\tau}G^{-1} \\ -G^{-1}F & G^{-1} \end{pmatrix}.$$

Without loss of generality one may require that the set of functions $\{\eta_m^n(\mathbf{r})\}$ satisfy the orthogonality conditions $G_{mm'}^{nn'} = \delta_m^{m'}\delta_n^{n'}$. These requirements are satisfied, for example, by the system

$$\eta_m^n(\mathbf{r}) = \psi_m^n(r)\psi_m(\varphi), \qquad (8)$$

where

$$\psi_m(\varphi) = \frac{1}{\sqrt{\pi(1+\delta_m^0)}} \cos\left[m\frac{\pi}{2} + \varphi E\left(\frac{m+1}{2}\right)\right],$$

$$\psi_m^n(\mathbf{r}) = \begin{cases} \dfrac{\mu_n^{(m)}}{R(z,\varphi)}\sqrt{\dfrac{2}{(\mu_n^{(m)})^2 - m^2}}\,\dfrac{J_m(\mu_n^{(m)}\rho/R)}{J_m(\mu_n^{(m)})}, & \text{for problem (1)—(2}_1\text{);} \\ \dfrac{\sqrt{2}}{R(z,\varphi)}\,\dfrac{J_m(\nu_n^{(m)}\rho/R)}{J_{m+1}(\nu_n^{(m)})}, & \text{for problem (1)—(2}_2\text{);} \end{cases}$$

$E(x)$ is the greatest integer not exceeding x; $\mu_n^{(m)}$ and $\nu_n^{(m)}$ are the n-th positive roots of the equations $\frac{d}{dx}J_m(x) = 0$ and $J_m(x) = 0$, respectively, and, in accordance with the adopted system of counting, the least roots are assigned the index $n = 0$.

Let us consider characteristic special cases.

Suppose that the shape of the boundary and the function $\alpha(\Pi)$ do not depend on the coordinate z. Then $R = R(\varphi)$ (the boundary is the surface of a cylinder with a generatrix of general form), $\alpha = \alpha(\varphi)$, the matrix H is constant (does not depend on z), and the matrix $F = 0$. An arbitrary solution of (7) is expressed by the formula

$$\mathbf{h} = \exp(Az)\mathbf{h}_0, \qquad \text{where } A = J^{-1}H = \begin{pmatrix} 0 & I \\ -\Phi & 0 \end{pmatrix}.$$

Taking into account the specific form of the matrix A, we may write

$$\mathbf{h} = \begin{pmatrix} \cos(\Phi^{1/2}z) & \Phi^{-1/2}\sin(\Phi^{1/2}z) \\ -\Phi^{1/2}\sin(\Phi^{1/2}z) & \cos(\Phi^{1/2}z) \end{pmatrix}\mathbf{h}_0,$$

whence

$$\mathbf{f} = \cos(\Phi^{1/2}z)\mathbf{f}_0 + \Phi^{-1/2}\sin(\Phi^{1/2}z)\mathbf{g}_0. \tag{9}$$

Putting $\Phi = P^{-1}\Phi_0 P$, where Φ_0 is real and diagonal and P is real and nonsingular, after substitution in (9) we obtain the arbitrary solution in the form of a combination of the eigenwaves of the waveguide under consideration, varying harmonically with the coordinate z:

$$P\mathbf{f} = \cos(\Phi_0^{1/2}z)P\mathbf{f}_0 + \Phi_0^{-1/2}\sin(\Phi_0^{1/2}z)P\mathbf{g}_0. \tag{10}$$

As we see, the propagation constants of the eigenwaves are equal to the elements of the matrix $\Phi_0^{1/2}$ and may be determined directly from the matrix $\Phi^{1/2}$ as its eigenvalues.

If, on the other hand, the shape of the boundary and $\alpha(\Pi)$ do not depend on the azimuth φ, $R = R(z)$, and the coefficients $G_{mm'}^{nn'}$, $F_{mm'}^{nn'}$, and $\Phi_{mm'}^{nn'}$ are proportional to $\delta_m^{m'}$. This leads to the splitting of the system (5) into independent subsystems corresponding to different values of the index m, which testifies to the independence of waves with different values of the index of azimuthal variations m. Considering such subsystems separately, one should take $M = 1$, while m is arbitrary but fixed.

As an example, consider the case $R = a = \text{const}$, $\alpha = \text{const}$. Selecting, in accordance with the exposition, the subsystem with index $m = 0$, and calculating $\Phi_{00}^{nn'} = [k^2 - (\mu_n^0/a)^2]\delta_n^{n'} - 2\alpha/a$ for problem (1)–(2_1) and determining the spectrum of the matrix $\Phi^{1/2}$, we obtain, to within $O(\alpha^2)$, the following values of the propagation constants:

$$h_n^{(0)} = \sqrt{k^2 - (\mu_n^{(0)}/a^2) - 2\alpha/a},$$

which, for $\alpha = 0$, pass into the well-known expressions for a circular cylindrical waveguide with rigid walls.

In the case where the boundary has period D along the z-axis, $H(z+D) = H(z)$, and the system (7) possesses all the properties set forth in [2]. Application of the formulas of the theory of parametric resonance given in [2] makes it possible to determine effectively the passbands and stopbands of a periodic waveguide as a function of the boundary shape and of the various parameters characterizing the problem.

As an example, let us present the results of a calculation for azimuthally symmetric waves ($m = 0$) in an azimuthally symmetric waveguide with periodic boundary $\rho = R(z) = R(z + D)$. Setting $R(z) = a[1 + \varkappa g(\vartheta z)]$, $\vartheta = 2\pi/D$, $\alpha = 0$ in (2_1),

$$g(\xi) = \sum_{l=1}^{\infty} (\alpha_l \cos l\vartheta z + \beta_l \sin l\vartheta z)\,, \quad \int_0^{2\pi} g(\xi)\, d\xi = 0,$$

we obtain a system of inequalities determining the stop bands in the $(\varkappa, \vartheta)$-plane:

$$\vartheta_{jh}^{(l)} - \varkappa\chi_{jh}^{(l)} + O(\varkappa^2) < \vartheta < \vartheta_{jh}^{(l)} + \varkappa\chi_{jh}^{(l)} + O(\varkappa^2) \tag{11}$$

$$j, h = 0, 1, 2, \dots; \quad l = 1, 2, \dots$$

Here

$$\vartheta_{jh}^{(l)} = (\omega_j + \omega_h)/l, \quad \omega_j^2 = k^2 - (\mu_j^{(0)}/a)^2;$$

$$\chi_{jh}^{(l)} = \begin{cases} \dfrac{\sqrt{\alpha_l^2 + \beta_l^2}}{\sqrt{\omega_j \omega_h}} \dfrac{\omega_j + \omega_h}{2l} \left| \omega_j \nu_{jh} + \omega_h \nu_{hj} \right|, & j \neq h, \\ \dfrac{\sqrt{\alpha_l^2 + \beta_l^2}}{l} \left| \dfrac{1}{\omega_j} \left[2k^2 - \left(\dfrac{\mu_j^{(0)}}{a} \right)^2 \right] - 2\omega_j \nu_{jj} \right|, & j = h; \end{cases}$$

$$\nu_{jh} = \frac{2(\mu_h^{(0)})^2}{(\mu_j^{(0)})^2 - (\mu_h^{(0)})^2} \quad \text{for } j \neq h, \qquad \nu_{jj} = -\frac{1}{(\mu_j^{(0)})^2}. \tag{12}$$

Fixing the index j in (11), we obtain the totality of all stop bands of the principal ($h = j$) and combination ($h \neq j$) resonances for each value $l = 1, 2, \ldots$ for the proper (j-th) waveguide wave satisfying Floquet' s condition.

Now consider a characteristic example of a nonsmooth boundary. Let us investigate what the conditions must be in the plane $z = z^*$ ($z_0 < z^* < z_1$) for joining two smooth waveguides. Let $S_+(S_-)$ be the domain bounded by the curve $\rho = R(z^* + 0, \varphi) = R_+(z^*, \varphi)$ (or, respectively, by the curve $\rho = R(z^* - 0, \varphi) = R_-(z^*, \varphi)$) and lying in the plane $z = z^*$; C be the intersection of S_+ and S_-; $C_+(C_-)$ be the set of points of $S_+(S_-)$ not belonging to $S_-(S_+)$. It is assumed that C is a nonempty connected set containing the trace of the z-axis in the plane $z = z^*$, and the union Π^* of the sets C_+ and C_- together with the boundary is contained in Π, while no other point of the plane $z = z^*$ belongs to Π.

Allowing the possibility of discontinuity of the values of the vectors $\mathbf{f}(z)$ and $\mathbf{g}(z)$ at the point $z = z^*$, it is not difficult to obtain, from considerations of extremality of the functional (6), with the aid of the formula for the total variation, the conditions at the point of discontinuity:

$$\mathbf{g}(z^* - 0) + G(C_-)\mathbf{f}(z^* - 0) = 0,$$

$$-\mathbf{g}(z^* + 0) + G(C_+)\mathbf{f}(z^* + 0) = 0.$$

Here

$$G(C_\pm) = \{G_{mm'}^{nn'}(C_\pm)\}, \quad G_{mm'}^{nn'}(C_\pm) = \iint_{C_\pm} \left(\eta_m^n \eta_{m'}^{n'} \right) \big|_{z = z^* \pm 0} \, dS.$$

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.