GROUPS OF GENERALIZED MOTIONS

1969

SovietRxiv

Abstract

Full Text

UDC 519.46

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(Presented by Academician A. D. Aleksandrov, 15 I 1969)

The present work is devoted to the study of groups of generalized motions in Riemannian spaces (^{1}, ^{2}). An analogue of the Killing equations is obtained for arbitrary groups of generalized motions. A connection is established between the set of all spaces conformal to a given Riemannian space and the conformal group in this space. All assertions in the paper are local in character.

1. Defects of invariance

The fundamental working concept in this paper will be the defect of invariance of a manifold with respect to a continuous group of transformations, introduced by L. V. Ovsyannikov (^{3}, ^{4}). Therefore, below we give the basic facts connected with the defect of invariance.

Let H denote a continuous group of transformations (in what follows we shall simply say the group H) of an n-dimensional Euclidean space $E(x)$ of points $x = (x^1, \dots, x^n)$ into itself. The element of the group H corresponding to the group parameter $a = (a^1, \dots, a^r)$ will be denoted by T_a; the transformation will be written in the form

$$x'^i = x'^i(x, a) \qquad (i = 1, \dots, n). \tag{1}$$

We shall write the basic infinitesimal operators of the group H in the form

$$X_\alpha = \xi^i_\alpha(x) \frac{\partial}{\partial x^i} \qquad (\alpha = 1, \dots, r). \tag{2}$$

Consider a manifold $\mathfrak{N} \subset E(x)$ of dimension $\dim \mathfrak{N} = n - s$, regularly defined (^{3}) by a system of equations

$$\psi^\sigma(x) = 0 \qquad (\sigma = 1, \dots, s), \tag{3}$$

and introduce the notation

$$T_a(\mathfrak{N}) = \bigcup_{x\in\mathfrak{N}} T_a x, \qquad H(\mathfrak{N}) = \bigcup_{T_a\in H} T_a(\mathfrak{N}). \tag{4}$$

If $\mathfrak{M} \subset E(x)$ is some manifold containing $\mathfrak{N}$ and possessing the property $H(\mathfrak{M}) = \mathfrak{M}$ (such a manifold, in particular, is $H(\mathfrak{N})$), then

$$H_{\mathfrak{N}} \subset \mathfrak{M}. \tag{5}$$

Definition 1. The nonnegative integer

$$\delta(\mathfrak{N}, H) = \dim H(\mathfrak{N}) - \dim \mathfrak{N} \tag{6}$$

is called the defect of invariance of $\mathfrak{N}$ with respect to the group H.

By $R(\|M(x)\|_{\mathfrak{N}})$ we shall denote the generic rank of the matrix $\|M(x)\|$, computed on the manifold $\mathfrak{N}$. A convenient way of computing the defect of invariance is given by the following important theorem (^{3}).

Theorem 1.

$$\delta(\mathfrak{N}, H) = R(\|X_\alpha \psi^\sigma(x)\|_{\mathfrak{N}}). \tag{7}$$

2. Riemannian spaces

For us the following point of view on an n-dimensional Riemannian space V_n will be convenient. Consider the Euclidean space $E(x, g)$ of variables x^i $(i = 1, \dots, n)$, g_{ij} $(i, j = 1, \dots, n)$, and all possible n-dimensional manifolds $\overline{V}_n \subset E(x, g)$, given by the equations

$$g_{ij} = \varphi_{ij}(x), \qquad (\det \|\varphi_{ij}\| \neq 0;\ \varphi_{ij} = \varphi_{ji};\ i, j = 1, \dots, n). \tag{8}$$

The manifold $\overline{V}'_n$, given by the equations $g_{ij} = \varphi'_{ij}(x)$, will be called equivalent to the manifold $\overline{V}_n$, and we shall write $\overline{V}'_n \sim \overline{V}_n$, if there exist functions $x'^i = x'^i(x)$ $(i = 1, \dots, n)$ satisfying the equations

$$\varphi'_{kl}(x'(x)) \frac{\partial x^k(x)}{\partial x^i} \frac{\partial x^l(x)}{\partial x^j} = \varphi_{ij}(x) \qquad (i, j = 1, \dots, n). \tag{9}$$

Then the class of all mutually equivalent manifolds $\overline{V}_n$ will define the Riemannian space V_n. We shall say that the manifold $\overline{V}_n$ specifies the space V_n in the coordinate system x^i, and the manifold $\overline{V}'_n \sim \overline{V}_n$—in the coordinate system x'^i $(i = 1, \dots, n)$. This method of specifying a Riemannian space coincides with the usual one (^{5}).

3. **Generalized motions.** Suppose a group H of transformations (1) is given, and suppose that the quantities g_{ij} $(i, j = 1, \dots, n)$ are transformed under it according to the formulas

$$g'_{ij} = g_{kl} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} \qquad (i, j = 1, \dots, n). \tag{10}$$

Then to the group H there will correspond a group $\overline{H}$ of transformations (1), (10) of the space $E(x, g)$ into itself. The expression for the infinitesimal operators of the group $\overline{H}$ is given in (^{1},^{2}). Take a manifold $\overline{V}_n$ specifying the space V_n in some coordinate system, and introduce the following

Definition 2. The number

$$\delta(V_n, H) = \delta(\overline{V}_n, \overline{H}) \tag{11}$$

is called the **defect of the Riemannian space** V_n with respect to the group H, and the group H is called a **group of generalized motions of defect** $\delta(V_n, H)$ in the Riemannian space V_n.

Denote by $\|\xi_{(\alpha)i,j} + \xi_{(\alpha)j,i}\|$ the matrix whose rows are numbered by the index α, and whose columns are numbered by the double index ij. Here the index α, for convenience, is written below, while the indices after the comma denote covariant differentiation in V_n. The following theorem gives an analogue of the Killing equations for groups of generalized motions.

Theorem 2.

$$\delta(V_n, H) = R(\|\xi_{(\alpha)i,j} + \xi_{(\alpha)j,i}\|). \tag{12}$$

Proof. Formula (12) follows from Theorem 1 if it is applied to the manifold $\overline{V}_n$ and the group $\overline{H}$.

Corollary. In order that the group H be a group of motions in the Riemannian space V_n, it is necessary and sufficient that the equality $\delta(V_n, H) = 0$ hold.

In the right-hand side of formula (11), as also of formula (12), a particular representative $\overline{V}_n$ of the space V_n explicitly enters. For Definition 2 to be correct, it is necessary to show that the number $\delta(V_n, H)$ does not depend on the choice of coordinate system in V_n, i.e., on the choice of $\overline{V}_n$.

Theorem 3. *The defect $\delta(V_n, H)$ does not depend on the choice of coordinate system in the space V_n.*

Proof. Under a transformation of coordinates in V_n, the columns of the matrix $\|\xi_{(\alpha)i,j} + \xi_{(\alpha)j,i}\|$ are transformed as components of a bivalent tensor. Therefore the columns of the transformed matrix will be linear combinations of the columns of the original matrix, and, consequently, the rank of the matrix does not change. This, in view of (12), proves the theorem.

4. **Invariant class of spaces and invariant spaces.** Let us clarify the meaning of the manifolds $\overline{T}_a(\overline{V}_n)$ and $\overline{H}(\overline{V}_n)$, constructed by formulas (4). The manifold $\overline{T}_a(\overline{V}_n)$ is specified by the equations

$$g_{ij} = \varphi_{kl}(x'(x,a))\frac{\partial x'^k(x,a)}{\partial x^i}\frac{\partial x'^l(x,a)}{\partial x^j} \qquad (i,j=1,\dots,n). \tag{13}$$

Therefore, for all $T_a \in H$,

$$\overline{T}_a(\overline{V}_n) \sim \overline{V}_n. \tag{14}$$

From the equality (see Definitions 1 and 2)

$$\dim \overline{H}(\overline{V}_n) = h + \delta(V_n, H) \tag{15}$$

and Theorem 3 it follows that the manifold $\overline{H}(\overline{V}_n)$ specifies, in a certain coordinate system, a certain class of Riemannian spaces depending on $\delta(V_n, H)$ arbitrary functions. We shall denote this class of spaces by $H(V_n)$.

Definition 3. The class of Riemannian spaces $H(V_n)$ is called the **invariant class of spaces** corresponding to the pair (V_n, H).

The construction of the invariant class $H(V_n)$, starting from the manifold $\overline{H}(\overline{V}_n)$, is carried out according to item 2. The basic property of the invariant class $H(V_n)$ consists in the fact that it is the smallest set of spaces invariant with respect to transformations of the group H that contains the space V_n. This follows from (5).

It follows from Definition 3 that there exist exactly $n(n+1)/2 - \delta(V_n, H)$ independent geometric quantities, identical for all spaces belonging to the class $H(V_n)$, at points with identical values of the coordinates x^i $(i=1,\dots,n)$. Taking, in particular, the space V_n and taking (14) into account, we obtain that there exist $n(n+1)/2 - \delta(V_n, H)$ independent geometric elements in the space V_n, invariant with respect to transformations of the group H, which we shall call **invariants of the space** V_n with respect to the group H.

5. The conformal group and conformal spaces. Consider, as the group of generalized motions, the group of conformal transformations in the space V_n. In doing so, an interesting connection will be established between the conformal group and conformal spaces (Theorem 6), which until now have been considered independently.

The space $V_n^{(\sigma)}$ with metric tensor

$$g_{ij} = \sigma(x)\varphi_{ij}(x) \qquad (i,j = 1,\dots,n) \tag{16}$$

is called conformal to the space V_n with metric tensor (8). Taking $\sigma(x)$ to be an arbitrary function, we obtain the entire class of spaces conformal to the given space V_n. We shall denote this class of spaces by $\{V_n^{(\sigma)}\}$. It is specified by the manifold $\{\overline{V}^{(\sigma)}\} = \bigcup_\sigma \overline{V}_n^{(\sigma)}$.

Definition 4. The group H is called a **group of conformal transformations** in the space V_n, if for every $T_a \in H$

$$\overline{T}_a(\overline{V}_n) \subset \{\overline{V}_n^{(\sigma)}\}. \tag{17}$$

In what follows we shall consider spaces V_n for which the conformal group is broader than the group of motions.

Theorem 4. If H is a group of conformal transformations in V_n, then

$$\delta(V_n, H) = 1. \tag{18}$$

Proof. The coordinates of the operators (2) of the group H satisfy equations (5)

$$\xi^i_{\alpha,j} + \xi^j_{\alpha,i} = \mu_\alpha(x)\varphi_{ij}(x) \qquad (i,j = 1,\dots,n;\ \alpha = 1,\dots,r). \tag{19}$$

Therefore from (12) we obtain

$$\delta(V_n, H) = R(\|\mu_\alpha(x)\varphi_{ij}(x)\|) = 1. \tag{20}$$

Theorem 5. The group H of conformal transformations in the space V_n is a conformal group in any space $V_n^{(\sigma)}$.

Proof. Condition (17), in view of (13), means that

$$\varphi_{kl}(x'(x,a))\frac{\partial {x'}^k(x,a)}{\partial x^i}\frac{\partial {x'}^l(x,a)}{\partial x^j} = f(x,a)\varphi_{ij}(x) \qquad (i,j = 1,\dots,n). \tag{21}$$

Multiplying both sides of (21) by the function $\sigma(x'(x,a))$ and setting $f_\sigma(x,a) = \dfrac{\sigma(x'(x,a))}{\sigma(x)} f(x,a)$, we obtain the equality

$$[\sigma(x'(x,a))\varphi_{kl}(x'(x,a))]\frac{\partial {x'}^k}{\partial x^i}\frac{\partial {x'}^l}{\partial x^j} = f_\sigma(x,a)[\sigma(x)\varphi_{ij}(x)] \qquad (i,j = 1,\dots,n), \tag{22}$$

whence it follows that $\overline{T}_a(\overline{V}_n^{(\sigma)}) \subset \{\overline{V}_n^{(\sigma)}\}$. The theorem is proved.

Theorem 6. Let H be a conformal group in V_n. Then

$$H(V_n) = \{V_n^\sigma\}. \tag{23}$$

Proof. It follows from Theorem 5 that $\bar{H}(\{\bar{V}_n^{(\sigma)}\}) = \{\bar{V}_n^{(\sigma)}\}$. Therefore, by virtue of (5), we shall have $\bar{H}(\bar{V}_n) \subset \{\bar{V}_n^{(\sigma)}\}$, whence, taking into account the equality $\dim\{\bar{V}_n^{(\sigma)}\} = \dim \bar{H}(\bar{V}_n) = n+1$, we obtain (by the local nature of the consideration) that $\bar{H}(\bar{V}_n) = \{\bar{V}_n^{(\sigma)}\}$. This also means that equality (23) holds. The theorem is proved.

6. Example. Consider a flat space S_n with metric tensor given by the equations $g_{ij} = \delta_{ij}$ $(i, j = 1, \dots, n)$, and an infinite group H with the basis infinitesimal operators

$$X_\mu = \frac{\partial}{\partial x^\mu}, \qquad X_{\nu\mu} = x^\mu \frac{\partial}{\partial x^\nu} - x^\nu \frac{\partial}{\partial x^\mu} \quad (\nu < \mu); \qquad \nu, \mu = 1, \dots, n-1.$$

$$X_f = f(x^n)\frac{\partial}{\partial x^n}, \qquad f(x^n) \text{ is an arbitrary function.}$$

It is not difficult to establish that $\delta(S_n, H) = 1$. The invariant class $H(S_n)$ is given in the present coordinate system by the equations

$$g_{\mu i} = \delta_{\mu i} \qquad (\mu = 1, \dots, n-1;\ i = 1, \dots, n),$$

$$g_{nn} = F, \qquad F = F(x^1, \dots, x^n) \text{ is an arbitrary function.}$$

Let us find the invariants of the space S_n with respect to the group H. For this purpose consider an elementary n-hedron, which is determined by specifying all $n(n+1)/2$ edges. Quantities identical for all spaces of the class $H(S_n)$ are the lengths of the $(n-1)n/2$ edges of the n-hedron lying in the $(n-1)$-dimensional subspace of the variables $(x^1, \dots, x^{n-1})$, and $n-1$ ratios of the lengths of the remaining n edges of the n-hedron. Thus the total number of identical quantities is $n(n+1)/2 - 1$. These same quantities are invariants of the space S_n with respect to the group H.

If, instead of an n-hedron, we take an elementary n-dimensional ball in S_n, then under transformations of the group H this ball will be stretched along the axis x^n, preserving all other dimensions.

It should be noted that, unlike the number of invariants, which does not depend on the choice of a coordinate system in the space, the concrete form of these invariants depends on the choice of the coordinate system.

7. Application to the Einstein equations. One of the difficulties in investigating the Einstein equations in the general theory of relativity is the large number of unknown functions—10 components of the metric tensor of the space V_4. To simplify these equations one may, for example, restrict oneself to the class of conformally flat spaces. This reduces the number of unknown functions to one; moreover, the resulting solutions admit a good physical interpretation ([6]).

Theorem 6 indicates how, in an analogous way, one can use invariant classes of spaces of various pairs (V_4, H). Namely, if one seeks a solution of the Einstein equations belonging to the class $H(V_4)$, then the number of unknown functions will be equal to $\delta(V_4, H)$. In this case, the problem of simplifying the Einstein equations is reduced to the problem of a physically reasonable choice of the initial space V_4 and of a group of generalized motions in this space with a sufficiently small defect.

The appearance of the present work was greatly aided by conversations with L. V. Ovsyannikov, which were not limited to discussion of the results obtained; the positive criticism of A. D. Aleksandrov was also very valuable. The author expresses his sincere gratitude to them.

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Received
6.I.1969

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Note: Figure translations are in progress. See original paper for figures.