ITERATIVE TWO-LAYER SCHEMES FOR NON-SELF-ADJOINT EQUATIONS

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Abstract

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ITERATIVE TWO-LAYER SCHEMES FOR NON-SELF-ADJOINT EQUATIONS

By the methods of papers ([1–3]), two-layer (one-step) iterative schemes are investigated for solving the equation $Au = f$, where A is a linear non-self-adjoint operator in a Hilbert space (H.s.). Implicit stationary iterative schemes are reduced to explicit schemes, for which problems on the minimum of the norm of the transition operator are solved, and values are obtained for the iterative parameters as functions of the amount of information about the operator A. Estimates are found for the convergence rate of the method of minimal corrections for the case $A \neq A^*$. The main attention is devoted to the choice of iterative parameters.

1. Let H be a real H.s., (,) the scalar product, and $\|x\| = \sqrt{(x,x)}$ the norm in H. Throughout we shall consider operators A, B, R, etc., mapping H into H. The notation is the same as in ([1]).

Consider the equation of the first kind

$$Au = f, \tag{1}$$

where A is a positive ($A > 0$) non-self-adjoint operator ($A \neq A^*$); u is the unknown, and f is a given vector in H. To solve equation (1) we shall use the two-layer iterative scheme

$$B(y_{k+1} - y_k)/\tau + Ay_k = f, \quad k = 0, 1, \dots,$$
$$\text{an arbitrary vector } y_0 \in H \text{ is given,} \tag{2}$$

where k is the iteration number, and B is a self-adjoint positive operator

$$B = B^* > 0. \tag{3}$$

All arguments are carried out under the assumption that the initial approximation y_0 is an arbitrary vector from H.

The implicit scheme (2), according to (1), is equivalent to the explicit scheme

$$(x_{k+1} - x_k)/\tau + Cx_k = \varphi, \quad k = 0, 1, 2, \dots, \quad x_0 = B^{1/2} y_0 \in H \tag{4}$$

for $x_k = B^{1/2} y_k$, $C = B^{-1/2} A B^{-1/2}$, $\varphi = B^{-1/2} f$, so that

$$\|x_k\| = \|y_k\|_B = \sqrt{(By_k, y_k)}.$$

Scheme (4) obviously corresponds to the equation $Cv = \varphi$, where

$$v = B^{1/2} u, \quad C = B^{-1/2} A B^{-1/2}, \quad \varphi = B^{-1/2} f. \tag{5}$$

To estimate the convergence rate of the iterations, consider the homogeneous equation with arbitrary $x_0 \in H$

$$x_{k+1} = Sx_k, \quad k = 0, 1, \dots, \quad S = E - \tau C, \tag{6}$$

where S is the transition operator of the explicit scheme and E is the identity operator. Equation (6) gives

$$\|x_{k+1}\| \leq \|S\| \|x_k\|, \quad \|x_n\| \leq \|S^n\| \|x_0\| \leq \|S\|^n \|x_0\|.$$

From the preceding it is clear that it is sufficient to restrict oneself to the study of the explicit scheme, to obtain for it an estimate of $\|S\|$, and to choose from the condition of minimizing $\|S\|$. The choice of the parameter τ and the estimate of the rate of convergence of the iterations are the main task of the theory.

2. Let us first consider the case when lower bounds are given for the operators C and C^{-1}:

$$C \geq \gamma_1 E \quad \text{or} \quad (Cx, x) \geq \gamma_1 \|x\|^2, \quad \gamma_1 > 0 \quad \text{for all } x \in H, \tag{7}$$

$$C^{-1} \geq \frac{1}{\gamma_2} E \quad \text{or} \quad \|Cx\|^2 \leq \gamma_2 (Cx, x), \quad \gamma_2 > 0 \quad \text{for all } x \in H. \tag{8}$$

From (3) it follows directly that

$$\|S\| = \|E - \tau C\| \leq \sqrt{1 - \xi}, \qquad \xi = \gamma_1/\gamma_2 \quad \text{for} \quad \tau = 1/\gamma_2. \tag{9}$$

For the transition operator

$$S = (E + \omega A)^{-1}(E - \omega A) = E - \tau C,$$

$$\tau = 2\omega, \quad C = (E+\omega A)^{-1} A = (A^{-1}+\omega E)^{-1}, \quad A \geq \delta E, \quad A^{-1} \geq \frac{1}{\Delta} E, \quad \Delta > \delta > 0,$$

in (3) the estimate was obtained

$$\|S\| \le \sqrt{(1-\sqrt{\eta})/(1+\sqrt{\eta})}, \qquad \eta = \delta/\Delta \quad \text{for } \omega = 1/\sqrt{\delta\Delta}, \quad \tau = 2\omega. \tag{10}$$

Using (9) for $C = (A^{-1} + \omega E)^{-1}$ and taking into account that $\gamma_1 = \delta/(1+\omega\delta)$, $\gamma_2 = \Delta/(1+\omega\Delta)$, we find $\omega = 1/\Delta$ and $\|S\| \le [(1-\eta)/(1+\eta)]^{1/2}$. Comparison with (10) shows that (9) is too crude an estimate (it is unfortunately not possible to improve (9) under conditions (7), (8)).

Remark. For the implicit scheme (2), conditions (7) and (8) are equivalent to the conditions

$$(Ax, x) \ge \gamma_1(Bx, x), \qquad (B^{-1}Ax, Ax) \le \gamma_2(Ax, x) \quad \text{for all } x \in H. \tag{11}$$

3. The amount of information concerning C (A and B) can be increased by specifying three numbers $\gamma_1, \gamma_2, \gamma_3$ instead of two. In this case the estimate of $\|S\|$ is improved and becomes the estimate for $\|S\|$ when $C = C^*$. In this section we consider the case of a complex Hilbert space $\hat{H}$.

Theorem 1. Let $\hat{H}$ be a complex Hilbert space with scalar product (,) and norm $\|x\| = \sqrt{(x,x)}$; let C be a linear operator from $\hat{H}$ into $\hat{H}$;

$$C = C_0 + iC_1, \quad C_0 = \operatorname{Re} C = \frac{1}{2}(C + C^*), \quad C_1 = \operatorname{Im} C = \frac{C - C^*}{2i}.$$

If the conditions

$$\gamma_1 E \le C_0 \le \gamma_2 E, \qquad \gamma_2 > \gamma_1 > 0, \qquad \|C_1\| \le \gamma_3, \qquad \gamma_3 \ge 0, \tag{12}$$

are satisfied, where $\gamma_2 > \gamma_1 > 0$, γ_3 are prescribed numbers, then for $\tau = \bar{\tau}$, equal to

$$\bar{\tau} = \tau_0(1-\varkappa^2)/(1+\varkappa\rho_0), \qquad \varkappa = \gamma_3/\sqrt{\gamma_1\gamma_2 + \gamma_3^2}, \qquad 0 \le \varkappa < 1,$$
$$\rho_0 = (\gamma_2 - \gamma_1)/(\gamma_2 + \gamma_1), \tag{13}$$

for the norm of the transition operator of scheme (6) the estimate

$$\|S\| = \|E - \bar{\tau}C\| \le \bar{\rho}, \qquad \text{where} \quad \bar{\rho} = (\rho_0 + \varkappa)/(1+\varkappa\rho_0) < 1 \tag{14}$$

is valid.

Theorem 2. Let $A = A_0 + iA_1$ and $B = B^* > 0$ be given on $\hat{H}$, and

$$\gamma_1 B \le A_0 \le \gamma_2 B, \qquad \gamma_2 > \gamma_1 > 0, \qquad |(A_1 y, y)| \le \gamma_3(By, y)$$
$$\text{for all } y \in \hat{H}. \tag{15}$$

Then for the solution of problem (2), for $\tau = \bar\tau$, the following a priori estimates hold:

$$\|y_n - u\|_B \le \bar\rho^n \|y_0 - u\|_B, \qquad \|y_{n+1} - y_n\|_B \le \bar\rho^n \|y_1 - y_0\|_B, \tag{16}$$

where $\bar\rho$ is given by formula (14).

This theorem refines the estimates of [5], where an approximate solution of the problem of finding $\inf\|S\|$ in the finite-dimensional case is given.

Lemma 1. Let A, R, B be given on $\hat H$, $R = R^* > 0$, $B = B^* > 0$,

$$\begin{gathered} c_1 R \le A_0 \le c_2 R, \qquad c_2 \ge c_1 > 0, \qquad |(A_1 y, y)| \le c_3 (Ry, y) \\ \text{for all } y \in \hat H, \\ c_3 \ge 0, \qquad \mathring\gamma_1 B \le R \le \mathring\gamma_2 B, \qquad \mathring\gamma_2 \ge \mathring\gamma_1 > 0. \end{gathered} \tag{17}$$

Then conditions (15) are satisfied with

$$\gamma_1 = c_1 \mathring\gamma_1, \qquad \gamma_2 = c_2 \mathring\gamma_2, \qquad \gamma_3 = c_3 \mathring\gamma_2.$$

4. Let H be a real Hilbert space, C be given on H,

$$C = C_0 + C_1,$$

where

$$C_0 = \frac{1}{2}(C + C^*), \qquad C_1 = \frac{1}{2}(C - C^*),$$

so that $(C_1 x, x) = 0$ for all $x \in H$.

If conditions (12) are satisfied, then estimate (14) is valid for the same values $\tau = \bar\tau$ and $\rho = \bar\rho$. The proof of this assertion practically coincides with the proof of Theorem 1.

Theorem 3. Let A, B be given on H, $B = B^* > 0$, $A = A_0 + A_1$, $A_0 = \frac{1}{2}(A - A^*)$, $A_1 = \frac{1}{2}(A - A^*)$, and suppose that the conditions

$$\gamma_1 B \le A_0 \le \gamma_2 B, \qquad (B^{-1} A_1 y, A_1 y) \le \gamma_3^2 (By, y), \tag{18}$$

where $\gamma_2 \ge \gamma_1 > 0$, $\gamma_3 \ge 0$ are prescribed numbers, are satisfied. Then, for $\tau = \bar\tau$, estimates (16) are valid for scheme (2).

Lemma 2. Let $R = R^* > 0$, $B = B^* > 0$, $A = A_0 + A_1$, $A_0 = A_0^* > 0$, and let A, B, R be given on H,

$$c_1 R \le A_0 \le c_2 R, \qquad (R^{-1} A_1 y, A_1 y) \le c_3^2 (Ry, y) \quad \text{for all } y \in H. \tag{19}$$

Then conditions (18) are satisfied with the constants

$$\gamma_1 = c_1\dot{\gamma}_1, \qquad \gamma_2 = c_2\dot{\gamma}_2, \qquad \gamma_3 = c_3\dot{\gamma}_3 \qquad (c_2 \geq c_1 > 0,\ c_3 \geq 0).$$

Operators $A \neq A^*$ and R, for which one of the groups of conditions (7)–(8), (19) or (17) is satisfied, will be called energetically equivalent (en. equiv.) (cf. with (4), where the spectral equivalence conditions contain 4 constants). If $A = A^*$, then conditions (7)–(8), (19) or (17) pass into the conditions $c_1R \leq A \leq c_2R$.

5. The a priori estimates (9) and (16) make it possible to obtain an estimate of the number of iterations sufficient for finding, by scheme (2), an approximate solution of problem (1) with prescribed accuracy $\varepsilon > 0$ (see (2)). In the practical application of the theory one must choose the operator B and compute these constants (see (3-5)). As B one may take the factorized operators (2):

$$B = (E + \omega R_1)(E + \omega R_2) \quad \text{for } R_2 = R_1^* > 0, \quad R = R_1 + R_2; \tag{20}$$

$$\begin{gathered} B = (E + \omega_1 R_1)(E + \omega_2 R_2) \quad \text{for } R_1 = R_1^* > 0, \\ R_2 = R_2^* > 0, \quad R_1R_2 = R_2R_1. \end{gathered} \tag{21}$$

The constants $\dot{\gamma}_1, \dot{\gamma}_2$ and the parameters $\omega, \omega_1, \omega_2$ are found in (2). It remains to compute the constants c_1, c_2, c_3 in conditions (11)–(18) and (17), where $R = R_1 + R_2$, then to find $\gamma_1 = c_1\dot{\gamma}_1$, $\gamma_2 = c_2\dot{\gamma}_2$, $\gamma_3 = c_3\dot{\gamma}_2$, and to use Theorems 1, 2, 3.

6. The two-layer iterative scheme

$$B(y_{k+1} - y_k)/\tau_{k+1} + Ay_k = f, \qquad k = 0, 1, 2, \ldots, \tag{22}$$

can be treated as a method of corrections:

$$y_{k+1} = y_k - \tau_{k+1}w_k, \qquad w_k = B^{-1}r_k, \qquad r_k = Ay_k - f, \tag{23}$$

where w_k is the correction, r_k the residual for the k-th iteration.

If the constants $\gamma_1, \gamma_2, \gamma_3$ are not known a priori (or are computed too roughly), then, in order to compute the parameters τ_{k+1}, it is expedient to use the steepest descent method (6, 7),

$$\tau_{k+1} = (w_k, r_k)/(Aw_k, w_k) \tag{24}$$

or the method of minimal corrections, setting

$$\tau_{k+1} = (Aw_k, w_k)/(B^{-1}Aw_k, Aw_k) \tag{25}$$

(for the explicit scheme, $B = E$, it coincides with the method of minimal residuals proposed in (8)).

If $A = A^*$, then method (24) converges in H_A, and method (25) in H_{A^2}, with the same rate as scheme (2) for constant $\tau = \tau_0$, $\tau_0 = 2/(\gamma_1 + \gamma_2)$:

$$\|Ay_n - f\| \le \rho_0^n \|Ay_0 - f\|, \qquad \rho_0 = (1-\xi)/(1+\xi), \qquad \xi = \gamma_1/\gamma_2$$

(see (8)). In the case of nonsymmetric A, only the method of minimal corrections is applicable. It converges with the same rate as for constant $\tau = \bar{\tau}$.

Theorem 4. Suppose conditions (18) are satisfied. Then, for the method of minimal corrections (23), (25), the estimates

$$\|Ay_n - f\| \le \rho^n \|Ay_0 - f\| \quad \text{for } n = 1, 2, \dots, \tag{26}$$

are valid, where ρ is determined by formula (14).

In proving Theorem 4, the implicit scheme (22) is reduced to the explicit scheme $x_{k+1} = x_k - \tau_{k+1}(Cx_k - \varphi)$, $x_k = B^{1/2} y_k$, $\varphi = B^{1/2} f$, after which the following fundamental result is used.

Lemma 3. Let C be a non-self-adjoint operator with bounds $\gamma_2 \ge \gamma_1 > 0$, $\gamma_1 E \le C \le \gamma_2 E$, and suppose that there exist numbers $\tau_* > 0$ and $\rho_* \in (0, 1)$ satisfying the conditions

$$\gamma_1 \tau_* \le 1 - \rho_*^2 \le \gamma_2 \tau_*, \tag{27}$$

such that the estimate

$$\|E - \tau_* C\| \le \rho_* < 1 \tag{28}$$

is valid. Then the inequality

$$(Cx, x)^2 \ge (1 - \rho_*^2) \|Cx\|^2 \|x\|^2 \quad \text{for all } x \in H \tag{29}$$

holds.

The conditions (27) and (28), with $\tau_* = \bar{\tau}$, $\rho_* = \bar{\rho}$, are fulfilled under (18) by virtue of Theorem 3. Lemma 3 makes it possible, for the explicit method of steepest descent when $A = A^*$, to obtain the estimate

$$\|y_n - u\|_A \le \rho_0^n \|y_0 - u\|_A, \quad \text{if } \gamma_1 B \le A \le \gamma_2 B. \tag{30}$$

7. The correction w_k in (21) is the solution of the equation $Bw_k = r_k$. The operator B is specified either constructively (for example, B is the factorized operator (20) or (21)) or is constructed as the result of some computational process. This occurs in one variant of the correction method—the two-stage method ([4,5]). Let $R = R^*$ be positive definite with the operator A, i.e., let one of the groups of conditions (7)–(8) or (19) be satisfied when $A \neq A^*$, or the condition $c_1 R \leq A \leq c_2 R$ when $A = A^*$. To compute the corrections w_k, the equation

$$Rw = r_k, \quad r_k = Ay_k - f, \tag{31}$$

is solved either by a direct method (then $B = R$), or by an iterative method with resolving operator T_m and zero initial approximation $w^{(0)} = 0$. After m iterations we have $w^{(m)} - w = -T_{mw}$, where $\|T_m\| \leq q < 1$, and w is the exact solution of (31). Taking $w_k = w^{(m)}$ and substituting $w = (E - T_m)^{-1} w^{(m)}$ into (31), we obtain $Bw_k = r_k$, where $B = R(E - T_m)^{-1}$. If $T_m = T_m^*$, and the operators T_m and R commute, then $B = B^* > 0$ and $\gamma_1 = 1 - q$, $\gamma_2 = 1 + q$ (see ([4,5])).

To find the iterations $y_{k+1} = y_k - \tau_{k+1} w_k$, one may set:
1) $\tau_{k+1} = \tau_0$ when $A = A^*$; then $\gamma_1 = (1-q)c_1$, $\gamma_2 = (1+q)c_2$;
2) $\tau_{k+1} = \bar{\tau}$ when $A \neq A^*$; then $\gamma_1 = (1-q)c_1$, $\gamma_2 = (1+q)c_2$, $\gamma_3 = (1+q)c_3$.

If c_1, c_2, c_3 are unknown, then one can use the two-stage variational method: 1) when $A = A^*$, compute τ_{k+1} by formula (24); 2) when $A \neq A^*$, compute τ_{k+1} by formula (25). From (25) it is clear that the two-stage method of minimal corrections (23), (25) requires the successive solution, by means of inner iterations T_m, of two equations:

$$Rw = r_k, \quad w^{(0)} = 0, \quad w_k = w^{(m)}; \qquad Rv = Aw_k, \quad v^{(0)} = 0, \quad v_k = B^{-1}Aw_k = v^{(m)}.$$

Knowing w_k and $B^{-1}Aw_k = v_k = v^{(m)}$, from (25) we find τ_{k+1}. By virtue of Theorem 4, the estimate (26) holds, in which ρ is determined by formula (14).

By analogy with ([2]), using estimates for $\|S\|$, it is not difficult to obtain a priori estimates expressing the stability of the iterative schemes with respect to the right-hand side.

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Note: Figure translations are in progress. See original paper for figures.