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ESTIMATES OF A SURFACE POTENTIAL GENERATED BY A GENERALIZED SHIFT OPERATOR

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Inequalities for singular integral operators have found broad application in the modern theory of partial differential equations. The problem of obtaining a priori estimates of solutions of boundary-value problems is essentially reduced to estimates of potential-type integral operators and their derivatives. In estimating volume potentials in the metrics L_p, Calderón–Zygmund' s result is used (see also paper ([3])) on the boundedness of singular integral operators. Surface potentials, however, are estimated by analogous inequalities, but already in norms of fractional order.

In the present note we formulate our second basic estimate of a singular integral with a kernel depending on the generalized difference of arguments. The case of a volume potential generated by a generalized shift operator was studied earlier by the authors in paper ([1]).

Let E_{n+2}^{++} be the unbounded domain of the $(n+2)$-dimensional Euclidean space of points (x, y, t), $x = (x_1, \dots, x_n)$, $y > 0$, $t > 0$, and let, in addition, E_{n+1}^{+} denote the $(n+1)$-dimensional half-space of points (x, y).

Put, for $0 < \lambda < 1$,

$$\|u\|_{L_{p,k}(E_{n+2}^{++})} = \left(\int_{E_{n+2}^{++}} |u(x,y,t)|^p y^k \, dx \, dy \, dt \right)^{1/p}, \tag{1}$$

$$[\varphi]_{x_i,\lambda,L_{p,k}(E_{n+1}^{+})} = \left(\int_{E_{n+1}^{+}} y^k \, dx \, dy \int_0^\infty \frac{|\varphi(x', x_i + h, y) - \varphi(x,y)|^p}{h^{1+p\lambda}} \, dh \right)^{1/p}, \tag{2}$$

$$[\varphi]_{y,\lambda,L_{p,k}(E^+_{n+1})} = \left(\int_{E^+_{n+1}} y^k\,dx\,dy \int_0^\infty \frac{|T^h_y\varphi(x,y)-\varphi(x,y)|^p}{h^{1+p\lambda}}\,dh\right)^{1/p}. \quad (3)$$

For $\lambda = 1$ we set

$$[\varphi]_{x_i,1,L_{p,k}(E^+_{n+1})} =$$

$$= \left(\int_{E^+_{n+1}} y^k\,dx\,dy \int_0^\infty \frac{|\varphi(x',x_i+2h,y)-2\varphi(x',x_i+h,y)+\varphi(x,y)|^p}{h^{1+p}}\,dh\right)^{1/p}, \quad (4)$$

$$[\varphi]_{y,1,L_{p,k}(E^+_{n+1})} =$$

$$= \left(\int_{E^+_{n+1}} y^k\,dx\,dy \int_0^\infty \frac{|T^{2h}_y\varphi(x,y)-2T^h_y\varphi(x,y)+\varphi(x,y)|^p}{h^{1+p}}\,dh\right)^{1/p}. \quad (5)$$

In the case $1 < \lambda < 2$ we set

$$[\varphi]_{y,\lambda,L_{p,k}(E^+_{n+1})} = \left(\int_{E^+_{n+1}} y^k\,dx\,dy \int_0^\infty \frac{\left|\frac{\partial}{\partial h}T^h_y\varphi(x,y)\right|^p}{h^{1+(\lambda-1)p}}\,dh\right)^{1/p}, \quad (6)$$

where T^h_y is the generalized shift operator, defined by the formula

$$T^h_y\varphi(x,y) = \frac{\Gamma((k+1)/2)}{\Gamma(1/2)\Gamma(k)} \int_0^\infty \varphi\Big(x, \sqrt{y^2+h^2-2yh\cos\alpha}\Big)\sin^{k-1}\alpha\,d\alpha. \quad (7)$$

It is not hard to see that the integral (6) exists for every $1 \le p < \infty$. Let $l > 0$. Define the number l' by the formula

$$l' = \begin{cases} [l], & \text{if } l \text{ is not an integer,} \\ l-1, & \text{if } l \text{ is an integer.} \end{cases}$$

We now put, for $r = (l/2)'$,

$$|\varphi|_{l,L_{p,k}} = \sum_{i=1}^{n} [D_{x_i}^{l}\varphi]_{x_i,l-l',L_{p,k}} + [B_y^r\varphi]_{y,l-2r,L_{p,k}}. \tag{8}$$

Here and below B_y^r denotes iteration of the Bessel operator

$$B_y = \frac{\partial^2}{\partial y^2} + \frac{k}{y}\frac{\partial}{\partial y}, \qquad k > 0.$$

In the case $p = 2$ and $l > 0$, the norm (8) is equivalent to the norm

$$|\varphi|^{*}_{l,L_{2,k}} = \left(\int_{E^+_{n+1}} (|\xi|^2 + \eta^2)^l \left|\hat{\varphi}_{(k-1)/2}(\xi,\eta)\right|^2 \eta^k \, d\xi \, d\eta\right)^{1/2}, \tag{9}$$

where $\hat{\varphi}_{(k-1)/2}$ is the mixed Fourier–Bessel transform, introduced by one of the authors in [2].

Indeed, for the corresponding finite functions in $L_{2,k}$, in the case $1 < \lambda = l - 2(l/2)' < 2$, we have

$$\frac{\partial}{\partial h} T_y^h B_y^r \varphi(x,y) = C_k \int_{E^+_{n+1}} \eta^{2r+1} \hat{\varphi}_{(k-1)/2}(\xi,\eta) e^{ix\xi} j_{(k-1)/2}(\eta h) j_{(k-1)/2}(\eta y) \eta^k \, d\xi \, d\eta.$$

According to Parseval' s equality for the mixed Fourier–Bessel transform, we obtain

$$[B_y^r\varphi]_{y,\lambda,L_{p,k}} = \left[C_k \int_{E^+_{n+1}} \eta^{2(2r+1)} \left|\hat{\varphi}_{(k-1)/2}\right|^2 \left(\int_0^\infty \frac{|j'_{(k-1)/2}(\eta h)|^2}{h^{1+(\lambda-1)2}} \, dh\right) \eta^k \, d\xi \, d\eta\right]^{1/2}, \tag{10}$$

where $r = (l/2)'$, $\lambda = l - 2(l/2)'$.

Taking into account the behavior of the Bessel function $j_\nu(x)$ near zero, we find that the integral in parentheses in equality (10) converges and is equal to $C_1\eta^{2\lambda-2}$. In a known way, the first terms on the right-hand side of equality (8) are transformed. From (10) and the last remark, in this case the equivalence of the norms (8) and (9) follows.

In the case $\lambda = 1$ (l is an integer), we have, as above,

$$[B_y^r\varphi]_{y,1,L_{p,k}} =$$

$$= \left[C_k \int_{E^+_{n+1}} \eta^{2r} \left| \widehat{\varphi}_{(k-1)/2} \right|^2 \left(\int_0^\infty \frac{|j_{(k-1)/2}(2\eta h) - 2j_{(k-1)/2}(\eta h) + 1|^2}{h^3}\, dh \right) \eta^k \, d\xi \, d\eta \right]^{1/2}. \tag{11}$$

Making the change of variable of integration according to the formula $\eta h = z$ and taking into account the estimate for the function $j_{(k-1)/2}$, we find that the integral in (11) converges and is equal to the quantity $C_2 \eta^2$. In an analogous way, the equivalence of (8) and (9) is proved also in the case $0 < \lambda < 1$.

2. We formulate a theorem whose proof is analogous to the corresponding proof given in (4).

Theorem 1. Let

$$u(x, y, t) = \int_{E^+_{n+1}} T^h_y K(x - s, y, t)\varphi(\xi, \eta)\eta^k \, d\xi \, d\eta, \qquad t > 0,$$

where T^h_y is the generalized shift operator and the kernel K satisfies the inequality

$$|K| \leq C \left(\sum |x_i|^{r_i} + y^{r_{n+1}} + t^{r_{n+2}} \right)^{-s},$$

where all numbers $r_j \geq 1$ and the number

$$\mu = s - \sum \frac{1}{r_i} + \frac{k+1}{r_{n+1}} + \frac{1}{pr_{n+2}} > 0 \qquad (p > 1).$$

a) If, for some $i \leq n$,

$$\int_{-\infty}^{\infty} K(x, y, t)\, dx_i = 0$$

and the number $l_i = r_i \mu$ satisfies the condition $0 < l_i < 1$, then

$$\|u\|_{L_{p,k}(E^{++}_{n+2})} \leq C[\varphi]_{x_i, l_i, L_{p,k}(E^+_{n+1})}.$$

b) If

$$\int_0^\infty K(x, y, t) y^k \, dy = 0$$

and the number $l_{n+1} = r_{n+1}\mu$ satisfies the condition $0 < l_{n+1} < 1$, then

$$\|u\|_{L_{p,k}(E^{++}_{n+2})} \leq C[\varphi]_{y, l_{n+1}, L_{p,k}(E^+_{n+1})}.$$

c) If $r_1 = r_2 = ... = r_{n+1} = r$,

$$\int_{E_{n+1}^+} K(x,y,t) y^k \, dx \, dy = 0$$

and the number $l = r\mu$ satisfies the condition $0 < l < 1$, then

$$\|u\|_{L_{p,k}(E_{n+2}^{++})} \leq C[\varphi]_{l, L_{p,k}(E_{n+1}^+)}.$$

The following theorem plays an important role in the theory of boundary-value problems for partial differential operators containing the singular Bessel differential operator.

Theorem 2. Let

$$u(x,y,t) = \int_{E_{n+1}^+} T_y^\eta K_{j,q}(x-\xi, y, t) \varphi(\xi, \eta) \eta^k \, d\xi \, d\eta, \qquad t > 0,$$

where $K_{j,q}$ is the Poisson kernel defined by the formulas of the paper (5). If, for every positive integer l, the norm $[\varphi]_{l-1/p, L_{p,k}} < \infty$, then

$$\|D^s B_y^r u\|_{L_{p,k}(E_{n+2}^{++})} \leq C[\varphi]_{l-1/p, L_{p,k}(E_{n+1}^+)}, \tag{12}$$

where $s + 2r = l + m_j + q + n + k + 1$.

The proof of this theorem is based on the use of Theorem 1, and we shall only outline the main scheme of the proof, omitting the detailed calculations. Without loss of generality one may assume that φ is a cor-

corresponding finite and infinitely differentiable function, and that $l = 1$ or $l = 2$.

Let $l = 1$. From the estimates of the Poisson kernels (see the theorem from (5)) it follows that

$$\left| D^s B_y^r K_{j,q} \right| \leq C \left[|x|^2 + y^2 + t^2 \right]^{-(n+k+2)/2}.$$

With the aid of this inequality, the fulfillment of all the conditions of Theorem 1 is proved.

In the case $l = 2$, we represent the kernel $K_{j,q}$ in the form $K_{j,q} = \Delta_B K_{j,q}$ and write

$$D^s B_y^r u(x,y,t) = \sum_{i=1}^{n} \int D^s D_{\xi_i}^2 B_y^r T_y^\eta K_{j,q+2}(x-\xi,y,t)\varphi(\xi,\eta)\eta^k \, d\xi \, d\eta + \int D^s B_y^{r+1} T_y^\eta K_{j,q+2}(x-\xi,y,t)\varphi(\xi,\eta)\eta^k \, d\xi \, d\eta. \tag{13}$$

In the first term on the right-hand side of equality (13), integration by parts with respect to the variable ξ_i should be performed, transferring one derivative to the function φ. The resulting functions can again be estimated with the aid of Theorem 1. Using the self-adjoint form of writing the operator B_y, the last term in (13) can be written in yet another form. The required estimate is then obtained as a result of applying Hölder' s inequality to the last integral.

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