



Soviet-era science, translated into English

Reports of the Academy of Sciences of the USSR

MATHEMATICS

1969

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.88215>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1969. Volume 184, No. 1

UDC 513.88:513.83

MATHEMATICS

V. S. RETAKH

ON THE DUAL OF A SUBSPACE OF A COUNTABLE INDUCTIVE LIMIT

(Presented by Academician A. N. Kolmogorov on 26 IV 1968)

Let $E = \varinjlim E_n$, where E_n are locally convex spaces, and let H be a vector subspace of E , endowed with the topology induced from E . There is a natural embedding

$$i : \varinjlim (E_n \cap H) \rightarrow H.$$

The question is posed: when is

$$i' : H' \rightarrow (\varinjlim (E_n \cap H))'$$

bijjective?

The corollary of Proposition 1 and Proposition 2 show the connection of this question with the surjectivity of the adjoint operator.

Proposition 1. Let $E = \varinjlim E_n$, $F = \varinjlim F_n$, let E and F be separable, and let E_n and F_n be Fréchet spaces ($n = 1, 2, 3, \dots$). Let T be a continuous linear operator mapping F into E , with sequentially closed image H . In order that T be a weak homomorphism of F onto H , it is necessary and sufficient that i' be bijective.

Corollary. Under the same assumptions, in order that $T' : E' \rightarrow F'$ be surjective, it is necessary and sufficient that T be a monomorphism and that i' be bijective.

Proposition 2. Let F be a bornological space, E the strict inductive limit of an increasing sequence (E_n) of its closed subspaces, and let T be a continuous linear operator mapping F into E , with $H = TF$. If $T' : E' \rightarrow F'$ is surjective, then i' is bijective.

An inverse spectrum (X_n) , with maps α_n , is a sequence

$$\dots \xrightarrow{\alpha_n} X_n \xrightarrow{\alpha_{n-1}} X_{n-1} \xrightarrow{\alpha_{n-2}} \dots \xrightarrow{\alpha_2} X_2 \xrightarrow{\alpha_1} X_1$$

of locally convex spaces X_n and continuous linear maps α_n . A morphism of an inverse spectrum (X_n) with maps α_n into an inverse spectrum (Y_n) with maps β_n is a set of continuous linear maps $\gamma_n : X_n \rightarrow Y_n$ such that

$$\beta_n \gamma_{n+1} = \gamma_n \alpha_n \quad (n = 1, 2, 3, \dots).$$

Inverse spectra and their morphisms form a category. Dually, the category of direct spectra is defined. Consider the functor Pro from the category of inverse spectra to the category of linear spaces, assigning to each inverse spectrum its projective limit (without topology). The functor Pro is left exact; therefore one can speak of its derived functors $\text{Pro}^1, \text{Pro}^2, \dots$. These derived functors were computed in (2).

I is called an injective object of the category of locally convex spaces (TLC) if a continuous linear mapping into I of a subspace of any locally convex space M can be extended to a continuous linear mapping of M into I .

Denote by $\mathcal{L}(\cdot, A)$ the contravariant functor from the category TLC to the category of linear spaces, assigning to each object of TLC the linear space of its morphisms into a fixed object A of TLC . Then to the direct spectrum

$$\mathcal{E} : E_1 \rightarrow E_2 \rightarrow \dots \rightarrow E_n \rightarrow \dots$$

there corresponds
inverse spectrum

$$\mathcal{L}_A(\mathcal{E}) : \dots \rightarrow \mathcal{L}(E_n, A) \rightarrow \dots \rightarrow \mathcal{L}(E_2, A) \rightarrow \mathcal{L}(E_1, A).$$

With the aid of Theorem 11.2 from (2), the following is proved.

Lemma. Let $E = \varinjlim E_n$ and $\forall n \exists p(n)$: if $q \geq p(n)$, then the topologies induced in E_n from $E_{p(n)}$ and E_q coincide. Then $\text{Pro}^1 \mathcal{L}_I(\mathcal{E}) = 0$ for any injective object I .

Put $G = E/H$, $G_n = E_n/(E_n \cap H)$; let τ_n be the quotient topology in G_n , and let $\tau_{n,p}$ be the topology induced in G_n from G_{n+p} . Using the lemma and the fact that every locally convex space is isomorphic to a subspace of an injective object of the category TLC (2), it is proved that

Theorem 1. In order that $i : \varinjlim (E_n \cap H) \rightarrow H$ be an isomorphism, it is sufficient that

$$\forall n \exists p : q \geq p \Rightarrow \tau_{n,p} = \tau_{n,q}.$$

We shall denote by u_n^0 and u_n^p the continuous linear functionals, respectively, on (G_n, τ_n) and on $(G_n, \tau_{n,p})$.

Theorem 2. Let E_n be normed spaces and suppose the topologies induced in E_n from E_{n+1} and E_{n+2} coincide for all n . Denote by B_n the unit ball in E_n and by \widetilde{B}_n the projection of B_n in G_n . For bijectivity of i' it is necessary and sufficient that

$$\forall n \exists p : \forall u_n^p \forall q \geq p \forall \varepsilon > 0 \exists u_n^q : |u_n^p(x) - u_n^q(x)| \leq \varepsilon \forall x \in \widetilde{B}_n.$$

Proof. Consider the exact sequence of direct spectra $0 \rightarrow (E_n \cap H) \rightarrow (E_n) \rightarrow (G_n) \rightarrow 0$ and pass to the algebraically exact sequence of inverse spectra $0 \rightarrow (G'_n) \rightarrow (E'_n) \rightarrow ((E_n \cap H)') \rightarrow 0$. Denoting the inverse spectra from left to right by $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$, we apply the functor Pro. We obtain the canonical exact sequence for a left-exact functor and its derived functors:

$$0 \rightarrow \text{Pro } \mathcal{L}_1 \rightarrow \text{Pro } \mathcal{L}_2 \rightarrow \text{Pro } \mathcal{L}_3 \rightarrow \text{Pro}^1 \mathcal{L}_1 \rightarrow \text{Pro}^1 \mathcal{L}_2 \rightarrow \dots$$

By the lemma, $\text{Pro}^1 \mathcal{L}_2 = 0$. Between $\text{Pro } \mathcal{L}_1, \text{Pro } \mathcal{L}_2, \text{Pro } \mathcal{L}_3$ and, respectively, $G', E', (\lim_{\rightarrow} (E_n \cap H))'$, there are natural algebraic isomorphisms such that the following diagram is commutative:

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Pro } \mathcal{L}_1 & \rightarrow & \text{Pro } \mathcal{L}_2 & \rightarrow & \text{Pro } \mathcal{L}_3 & \rightarrow & \text{Pro}^1 \mathcal{L}_1 & \rightarrow & 0 \\ & & \parallel & & \parallel & & \parallel & & & & \\ 0 & \rightarrow & G' & \rightarrow & E' & \rightarrow & (\lim_{\rightarrow} (E_n \cap H))' & \rightarrow & & & 0 \\ & & \parallel & & \parallel & & i' \uparrow & & & & \\ 0 & \rightarrow & G' & \rightarrow & E' & \rightarrow & H' & \rightarrow & & & 0 \end{array}$$

Since the lower row is exact, the bijectivity of i' is equivalent to the equality $\text{Pro}^1 \mathcal{L}_1 = 0$. To complete the proof of the theorem it is enough to introduce in each G'_n the strong topology and apply Corollary 11.4 from (2).

We shall denote by \bar{u}_k the continuous linear functionals on E whose restriction to $E_k \cap H$ is zero, and by \bar{v} the continuous linear functionals on E whose restriction to H is zero.

Corollary. Let E be the strict inductive limit of a sequence (E_n) of normed subspaces. For bijectivity of i' it is necessary and sufficient that

$$\forall n \exists p : \forall \bar{u}_p \forall \varepsilon > 0 \exists \bar{v} : |\bar{u}_p(x) - \bar{v}(x)| \leq \varepsilon \forall x \in B_n.$$

A theorem analogous to Theorem 2 can also be proved in the case where the E_n have a countable fundamental system of bounded sets.

In the case where the E_n are arbitrary locally convex spaces, the following holds.

Theorem 3. Let $E = \varinjlim E_n$. For the bijectivity of i' it is sufficient that, for every n , there exist a neighborhood of zero U_n in E_n such that $U_n \subset U_{n+1}$,

$$\forall n \exists p : \forall u_n^p \forall q \geq p \forall \varepsilon > 0 \exists u_n^q : |u_n^p(x) - u_n^q(x)| \leq \varepsilon \quad \forall x \in \tilde{U}_n,$$

where \tilde{U}_n is the image of U_n in G_n .

Corollary 1. Let E be a strict inductive limit of a sequence (E_n) of its subspaces. For the bijectivity of i' it is sufficient that there exist a neighborhood of zero U in E such that

$$\forall n \exists p : \forall \bar{u}_p \forall \varepsilon > 0 \exists \bar{v} : |\bar{u}_p(x) - \bar{v}(x)| \leq \varepsilon \quad \forall x \in E_n \cap U.$$

Corollary 2. Let $E = \varinjlim E_n$, where the mappings of E_n into E_{n+1} are weakly completely continuous, and the E_n are separable ($n = 1, 2, \dots$). If $E_n \cap H$ is closed in E_n for all n , then i' is bijective.

In the case where the E_n are reflexive Banach spaces, the result of Corollary 2 was obtained by Foiaş and Marinescu^(3,4), and for closed H , by Komatsu⁽¹⁾.

I express my gratitude to V. P. Palamodov and D. A. Raikov for their attention to the work.

Moscow State Pedagogical Institute
named after V. I. Lenin

Received
25 IV 1968

REFERENCES

- ¹ H. Komatsu, J. Math. Soc. Japan, **19**, No. 3, 366 (1967).
- ² V. P. Palamodov, *Matem. sborn.*, **75**, issue 4, 567 (1968).
- ³ C. Foiaş, G. Marinescu, C. R., **261**, No. 23, 4958 (1965).
- ⁴ C. Foiaş, G. Marinescu, C. R., **263**, No. 12, 390 (1966).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.