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# GENERAL INTEGRAL REPRESENTATIONS

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## Abstract

## Full Text

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MATHEMATICS

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# GENERAL INTEGRAL REPRESENTATIONS

(Presented by Academician M. A. Lavrent'ev on 8 VII 1968)

The author (<sup>1-7</sup>), in the cases of one and of several complex variables, has established a number of general integral representations of holomorphic functions. In the present note new general integral representations of holomorphic functions of one and several complex variables are indicated. Thus, in the case  $n = 1$  (§ 2) a general integral representation (1) is given, which rather fully reflects the specific properties of star-shaped domains, while in the case  $n \geq 2$  (§ 3) a general integral representation (4) is given, which rather fully reflects the specific properties of bounded convex complete  $n$ -circular domains and whose other important feature is its close connection with the Cauchy formula of one complex variable. In the exposition we adhere to the notation and definitions used in (<sup>1-3,5-8</sup>).

§ 1. Let  $G$  be a star-shaped domain with respect to the origin in the space  $C^n$  of complex variables  $z_1, \dots, z_n$ ,  $n \geq 1$ , and let the function  $f = f(z_1, \dots, z_n)$  be holomorphic in  $G$ . Let, further,  $k$  and  $\tilde{k}$  be natural numbers, and let  $\gamma_1, \dots, \gamma_k$  and  $\tilde{\gamma}_1, \dots, \tilde{\gamma}_{\tilde{k}}$  be arbitrary positive numbers satisfying  $\gamma_j \geq 1$  ( $j = 1, \dots, k$ ) and  $\tilde{\gamma}_{\tilde{j}} \geq 1$  ( $\tilde{j} = 1, \dots, \tilde{k}$ ). The author (<sup>2,3,5</sup>), in the case of the domain  $G$ , introduced the operator

$$L_{\left(\begin{smallmatrix} \gamma_1 \\ \gamma_k \end{smallmatrix}\right)}^{(k)}[f],$$

which we shall now denote by  $L_{\gamma_1, \dots, \gamma_k}^{(k)}[f]$  or, briefly, by  $L_a^{(k)}[f]$  ( $a = (\gamma_1, \dots, \gamma_k)$ ), and found the inverse operator  $L_a^{(-k)}[f]$  (in the former notation,  $L_{\left(\begin{smallmatrix} \gamma_1 \\ \gamma_k \end{smallmatrix}\right)}^{(-k)}[f]$ ).

Denote  $\tilde{a} = (\tilde{\gamma}_1, \dots, \tilde{\gamma}_{\tilde{k}})$ . Let now  $k$  and  $\tilde{k}$  be nonnegative integers satisfying  $k + \tilde{k} > 0$ . Introduce the notation

$$L_{a\tilde{a}}^{(k, -\tilde{k})}[f] = L_{\tilde{a}}^{(-\tilde{k})}[L_a^{(k)}[f]], \quad L_{a\tilde{a}}^{(-k, \tilde{k})}[f] = L_{\tilde{a}}^{(\tilde{k})}[L_a^{(-k)}[f]]^*$$

and set  $L_{a\tilde{a}}^{(0,0)}[f] = f$ . It is not difficult to see that the first of these two new operators is inverse to the second, and conversely.

Let, for each  $j$  from the set  $\{1, \dots, k\}$  ( $k$  is a natural number),  $\delta_1^{(j)}, \dots, \delta_n^{(j)}$  be arbitrary nonnegative numbers satisfying  $\delta_1^{(j)} + \dots + \delta_n^{(j)} > 0$ . Analogously, let, for each  $\tilde{j}$  from the set  $\{1, \dots, \tilde{k}\}$  ( $\tilde{k}$  is a natural number),  $\tilde{\delta}_1^{(\tilde{j})}, \dots, \tilde{\delta}_n^{(\tilde{j})}$  be arbitrary nonnegative numbers satisfying  $\tilde{\delta}_1^{(\tilde{j})} + \dots + \tilde{\delta}_n^{(\tilde{j})} > 0$ . Suppose that the function  $f = f(z_1, \dots, z_n)$  is holomorphic in the complete  $n$ -circular ( $n \geq 2$ ) domain  $Q$  with center at the point  $(0, \dots, 0)$ . The author (7), in the case of the domain  $Q$ , introduced the operator  $L_A^{(k)}[f]$  (here  $A = (A_1, \dots, A_k)$ ,  $A_j = (\gamma_j, \delta_1^{(j)}, \dots, \delta_n^{(j)})$  ( $j = 1, \dots, k$ )) and found the inverse operator  $L_A^{(-k)}[f]$ . Denote  $\tilde{A}_{\tilde{j}} = (\tilde{\gamma}_{\tilde{j}}, \tilde{\delta}_1^{(\tilde{j})}, \dots, \tilde{\delta}_n^{(\tilde{j})})$  ( $\tilde{j} = 1, \dots$

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\* We set  $L_a^{(0)}[f] = f$ ,  $L_{\tilde{a}}^{(0)}[f] = f$ .

$\dots, \tilde{k}$ ),  $\tilde{A} = (\tilde{A}_1, \dots, \tilde{A}_{\tilde{k}})$ . Let now  $k$  and  $\tilde{k}$  be nonnegative integers with  $k + \tilde{k} > 0$ . Introduce the notation

$$L_{AA}^{(k, -\tilde{k})}[f] = L_A^{(-\tilde{k})}[L_A^{(k)}[f]], \quad L_{AA}^{(-k, \tilde{k})}[f] = L_A^{(\tilde{k})}[L_A^{(-k)}[f]]$$

and put

$$L_{AA}^{(0,0)}[f] = f.$$

It is not difficult to see that the first of these two new operators is inverse to the second, and conversely.

**Remark 1.** In the same way as above, in the case of a domain  $G$  two other groups of operators are introduced:

$$\Gamma_{aa}^{(k, -\tilde{k})}[\operatorname{Re} f], \quad \Gamma_{aa}^{(-k, \tilde{k})}[\operatorname{Re} f],$$

$$J_{aa}^{(k, -\tilde{k})}[\operatorname{Re} f], \quad J_{aa}^{(-k, \tilde{k})}[\operatorname{Re} f]$$

(the same with replacement of  $\operatorname{Re} f$  by  $\operatorname{Im} f$  and by  $f$ ). Analogously also in the case of a domain  $Q$ .\*\*

§ 2. Let  $G_1$  be a star-shaped domain with respect to the origin in the space  $C^1$ , bounded by a closed rectifiable Jordan curve  $\Gamma$ .

**Theorem 1.** *Let the function  $f(z)$  be holomorphic in the domain  $G_1$ , and let  $a$  be a number equal to 0 or 1. Then, if the function  $f^{(\alpha)}(z)$  and all its derivatives up to order  $\mu$  ( $\mu \geq 0$ ) inclusive are continuous in the closed domain  $\bar{G}_1$ , then for  $k = 0, 1, \dots, \mu$ ;  $\tilde{k} = 0, 1, 2, \dots$ ;  $z \in G_1$*

$$f(z) = af(0) + \frac{z^\alpha}{2\pi i} \int_{\Gamma} J_{1,a}^{(-\alpha)} \left[ L_{aa}^{(-k, \tilde{k})} \left[ \frac{1}{\xi - z} \right] \right] L_{aa}^{(k, -\tilde{k})} [f^{(\alpha)}(\xi)] d\xi, \quad (1)$$

where the integration is performed over the contour  $\Gamma$  in the positive direction.

In the proof one uses the properties of the author's operators for star-shaped domains\*\*\* and Cauchy's formula.

Among the many consequences following from formula (1), let us note one:

$$f(z)|_{z \in G_1} = \frac{\tilde{k}!}{2\pi i} \int_{\Gamma} \frac{\xi^{\tilde{k}}}{(\xi - z)^{\tilde{k}+1}} L_{1, \tilde{k}}^{(-\tilde{k})}[f(\xi)] d\xi \quad (\tilde{k} = 0, 1, 2, \dots).$$

For  $\tilde{k} = 0$  this formula becomes Cauchy's formula.

**Remark 2.** We note that if the domain  $G_1$  is the disk  $|z| < 1$ , then under the conditions of Theorem 1 for  $k = 0, 1, \dots, \mu$ ;  $\tilde{k} = 0, 1, 2, \dots$ ;  $|z| < 1$  the following formulas also hold:

$$f(z) = af(0) + \frac{z^\alpha}{2\pi} \int_0^{2\pi} J_{1,a}^{(-\alpha)} \left[ J_{aa}^{(-k, \tilde{k})} \left[ \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos(\varphi - \psi)} \right] \right] \times \\ \times L_{aa}^{(k, -\tilde{k})}[f^{(\alpha)}(e^{i\varphi})] d\varphi \quad (z = \rho e^{i\psi}), \quad (2)$$

$$f(z) = af(0) + iz^\alpha \operatorname{Im} f^{(\alpha)}(0) +$$

$$+ \frac{z^\alpha}{2\pi} \int_0^{2\pi} J_{1,a}^{(-\alpha)} \left[ L_{aa}^{(-k, \tilde{k})} \left[ \frac{e^{i\varphi} + z}{e^{i\varphi} - z} \right] \right] \operatorname{Re} L_{aa}^{(k, -\tilde{k})}[f^{(\alpha)}(e^{i\varphi})] d\varphi. \quad (3)$$

§ 3. Let us pass to the case  $n \geq 2$ .

**Theorem 2.** Let  $D \in (T)$ , the function  $f(z)$  ( $n \geq 2$ )\*\*\* be holomorphic in  $D$ , and let  $a$  be a number equal to 0 or 1. Then, if the functions  $f_\nu^{(\alpha)}(z)$ ,  $\nu = 1, \dots, n$ , and all their partial derivatives up to order  $\mu$  ( $\mu \geq 0$ ) inclusive are continuous in  $\bar{D}$ , then for  $k = 0, 1, \dots, \mu$ ;  $\tilde{k} = 0, 1, 2, \dots$ ;  $z \in D^*$

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\* We put  $L_A^{(0)}[f] = f$  (in (7) it was put  $L_A^{(0)}[f] = f$ ).

\*\* For the operators  $\Gamma_a^{(k)}[\operatorname{Re} f]$ ,  $\Gamma_a^{(-k)}[\operatorname{Re} f]$ ,  $J_a^{(k)}[\operatorname{Re} f]$ ,  $J_a^{(-k)}[\operatorname{Re} f]$  see (5), § 1. The operators  $\Gamma_A^{(k)}[\operatorname{Re} f]$ ,  $\Gamma_A^{(-k)}[\operatorname{Re} f]$ ,  $J_A^{(k)}[\operatorname{Re} f]$ ,  $J_A^{(-k)}[\operatorname{Re} f]$  are introduced analogously.

\*\*\* On these operators see (1-3,5) and § 1 of the present note.

\*\*\*\*  $z = (z_1, \dots, z_n)$  (and so everywhere below in § 3).

$$f(z) = \alpha f(0) + \frac{1}{n + \alpha(1 - n)} \sum_{\nu=1}^n \frac{z_\nu^\alpha}{(2\pi)^{n_i}} \int d\omega_\zeta \int d\omega_\theta \int_{|\xi|=1} L_{\alpha+1, n-1}^{(n-1-\alpha)} \left[ L_{AA}^{(-k, \tilde{k})} \left[ \frac{1}{\xi - u} \right] \right] \times \\ \times L_{AA}^{(k, -\tilde{k})} [F_{0, \nu}^{(\alpha)}(\xi, r, \theta)] d\xi, \quad (4)$$

where the circle  $|\xi| = 1$  is oriented in the usual way.

The proof is similar to the proof of Theorem 5.1 in the author's paper <sup>(5)</sup>.

**Remark 3.** From the integral representation (4) it is clear that it has the same important property (connection with the Cauchy integral) as the integral representations (2) previously obtained by the author <sup>(4,5)</sup> from <sup>(4)</sup> (see <sup>(4)</sup>, p. 1219) and (5.1) from <sup>(5)</sup> (see <sup>(5)</sup>, § 5, item 4)\*.

Among the many corollaries following from formula (4), let us note one:

$$f(z) = \frac{(n - k + \tilde{k} - 1)!}{(2\pi)^{n_i}} \int d\omega_\zeta \int d\omega_\theta \int_{|\xi|=1} \frac{\xi^{n-k+\tilde{k}-1}}{(\xi - u)^{n-k+\tilde{k}}} \times \\ \times L_{n-k, n-k+\tilde{k}-1}^{(-\tilde{k})} \left[ L_{n-k, n-1}^{(k)} [F_0(\xi, r, \theta)] \right] d\xi$$

$$(k = 0, 1, \dots, \mu \ (0 \leq \mu \leq n - 1); \ \tilde{k} = 0, 1, 2, \dots).$$

For  $\tilde{k} = 0$  this formula becomes the formula obtained earlier, in a somewhat different form, by Opyal and Sityak <sup>(8)</sup>.

**Remark 4.** With the aid of the new operators introduced in the present paper, other integral representations previously obtained by the author (<sup>(3)</sup>, Theorem 5; <sup>(7)</sup>, Theorems 2, 3) in the case of  $n$  ( $n \geq 2$ ) complex variables are also generalized. Thus, under the assumptions of Theorem 5 (in the case of convex domains  $D^{**}$ ) from <sup>(3)</sup>, along with formula (3) written in <sup>(6)</sup>, there is also a more general formula, obtained from the indicated formula (3) by replacing

$$L_{\left(\begin{smallmatrix} -k \\ \gamma_1 \\ \gamma_k \end{smallmatrix}\right)}^{(-k)} \left[ \left( \sum_{l=1}^n (\xi_l - z_l) \Phi'_{\xi_l} \right)^{-1} \right], \quad L_{\left(\begin{smallmatrix} k \\ \gamma_1 \\ \gamma_k \end{smallmatrix}\right)}^{(k)} [F_\nu^{(\alpha)}(\xi)]$$

respectively by

$$L_{\alpha\tilde{\alpha}}^{(-k, \tilde{k})} \left[ \left( \sum_{l=1}^n (\xi_l - z_l) \Phi'_{\xi_l} \right)^{-1} \right], \quad L_{\tilde{\alpha}\tilde{\alpha}}^{(k, -\tilde{k})} [F_\nu^{(\alpha)}(\xi)] \quad (\tilde{k} = 0, 1, 2, \dots).$$

Among the many corollaries following from this more general formula, let us note one:

$$F(z) = \frac{(n - k + \tilde{k} - 1)!}{(2\pi i)^n} \int_{\partial D} \frac{L_{n-k, n-k+\tilde{k}-1}^{(-\tilde{k})} [L_{n-k, n-1}^{(k)} [F(\xi)]] \sum_{l=1}^n \sigma[l] d\bar{\xi}[l] \wedge d\xi}{\lambda^{k-\tilde{k}} \left( \sum_{l=1}^n (\xi_l - z_l) \Phi'_{\xi_l} \right)^{n-k+\tilde{k}}}$$

$$(k = 0, 1, \dots, \mu \ (0 \leq \mu \leq n - 1); \ \tilde{k} = 0, 1, 2, \dots).$$

For  $\mu = 0$ ,  $\tilde{k} = 0$  this formula becomes the formula established earlier by L. A. Aizenberg <sup>(9)</sup>.

Under the assumptions of Theorem 2(3) from <sup>(7)</sup>, along with formula (2) (respectively (3)) from <sup>(7)</sup>, there will also be valid a more general formula obtained from

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\* Under the assumptions of Theorem 2, along with formula (4), there are also two general formulas (corresponding to formulas (2), (3)), one of which has the same property as formula (4), but with respect to the Poisson integral, the other with respect to the Schwarz integral.

\*\* By  $D$  is meant the same convex domain in  $C^n$  that is considered in <sup>(9,1)</sup>.

formula (2) (respectively, (3)) by replacing  $L_A^{(-k)}[I]$ ,  $L_A^{(k)}[f_\nu^{(\alpha)}(\xi)]$  (respectively,  $L_A^{(-k)}[J_p]$ ,  $L_A^{(k)}[f_\nu^{(\alpha)}(r_1(\tau)e^{i\varphi_1}, \dots, r_n(\tau)e^{i\varphi_n})]$ ) respectively by  $L_{AA}^{(-k, \tilde{k})}[I]$ ,  $L_{AA}^{(k, -\tilde{k})}[f_\nu^{(\alpha)}(\xi)]$  respectively  $L_{AA}^{(-k, \tilde{k})}[J_p]$ ,  $L_{AA}^{(k, -\tilde{k})}[f_\nu^{(\alpha)}(r_1(\tau)e^{i\varphi_1}, \dots, r_n(\tau)e^{i\varphi_n})]$ , where  $\tilde{k} = 0, 1, 2, \dots$ . Similarly for the corresponding formulas in the case  $p = 0$  (see (7), p. 264, footnote). We note that from the more general formula corresponding to formula (a) from (7) (see (7), p. 264, footnote), there follows, as a corollary, the formula

$$f(z) = \frac{\tilde{k}_1! \dots \tilde{k}_n!}{(2\pi i)^n} \int_{C_1} \frac{d\xi_1}{\xi_1} \dots \int_{C_n} \frac{L_A^{(-\tilde{k})}[f(\xi)]}{\left(1 - \frac{z_1}{\xi_1}\right)^{\tilde{k}_1+1} \dots \left(1 - \frac{z_n}{\xi_n}\right)^{\tilde{k}_n+1}} \frac{d\xi_n}{\xi_n}$$

(here  $\tilde{k} = \tilde{k}_1 + \dots + \tilde{k}_n$  ( $\tilde{k}_1, \dots, \tilde{k}_n$  are arbitrary nonnegative integers,  $L_A^{(-\tilde{k})}[f] = L_{1, \tilde{k}_n}^{(-\tilde{k}_n)} [L_{1, \tilde{k}_{n-1}}^{(-\tilde{k}_{n-1})} \dots [L_{1, \tilde{k}_1}^{(-\tilde{k}_1)} [f]] \dots]$ , and  $z$  is the same as in Theorem 2 of (7); moreover, for this formula Remark 2 from (7), including the footnote to it, remains in force (see (7), p. 265)). For  $\tilde{k} = 0$  this formula becomes the Cauchy formula.

**Remark 5.** Taking into account what was noted earlier (see, for example, (7), Remark 5), the remarks concerning the corresponding integrals remain fully valid for all the content of the present note connected with  $\gamma_1, \dots, \gamma_k, \tilde{\gamma}_1, \dots, \tilde{\gamma}_k$ , also in the case when  $\gamma_1, \dots, \gamma_k, \tilde{\gamma}_1, \dots, \tilde{\gamma}_k$  are arbitrary positive numbers.

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*Note: Figure translations are in progress. See original paper for figures.*

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