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SOLUTION OF A CLASS OF MAGNETOHYDRODYNAMIC PROBLEMS WITH MAGNETIC-FIELD AMPLIFICATION

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Many observational data on the Sun suggest that beneath the surface of the solar photosphere there is a large-scale (“general”) toroidal magnetic field, which changes its sign every 11 years. Therefore, for solar physics, models of oscillatory hydromagnetic dynamos are of interest. The most characteristic type of plasma motion in the outer part of the Sun is cellular convection of the Bénard type (matter rises at the center of each cell, then spreads out to the sides and descends along the edges). Deeper layers are convectively stable, and there, owing to the negligible ohmic dissipation, the magnetic field can hardly be substantially restructured within observable times. It is natural to expect that convection in the subphotospheric zone is largely responsible for the operation of the solar dynamo. Its possible scheme, in the most general terms, may be imagined as follows. The differential rotation of the Sun forms, from an existing weak meridional field, a stronger azimuthal (toroidal) field. Each convective cell, winding up the lines of force of the toroidal field, amplifies it still further, and the action of Coriolis forces leads to a rotation of the entire pattern about the axis of the cell. Diffusion of the magnetic field gradually smooths the inhomogeneities, owing to which a large-scale meridional field is generated, opposite in sign to the old one. After some time the meridional component of the general field changes its sign, and the whole process repeats.

For developing dynamo theories of this kind it is necessary to investigate what magnetic configurations are produced by the hydromagnetic “activity” of a convective cell when Coriolis forces act. In the present work this problem is solved for the simplest model. It is of interest that from the chosen special case it is not difficult to pass to various situations with similar velocity fields. They may also be of interest in applications, and therefore are discussed in the article.

In connection with the dynamo problem, Tverskoi ([1]) considered an axisymmetric vortex ring in which the velocity of the substance has no azimuthal component. The current lines are situated in the planes of meridional sections of the vortex and are circles with centers at a distance a from the axis. All lines of radius r form the surface of a circular torus. In this case it is convenient to use the orthogonal coordinate system r, φ, χ, where φ is the azimuthal angle and χ is the polar angle in the plane of the meridional section. The Lamé parameters are $h_r = 1,\ h_\varphi = a + r\cos\chi,\ h_\chi = r$.

Assuming that not only v_χ, but also v_φ, is nonzero, we obtain a schematic representation of a convective cell with a Coriolis perturbation of the velocity field. From the incompressibility condition $\operatorname{div}\mathbf{v} = 0$ we find $v_\chi = V(r)/[1 + (r/a)\cos\chi]$, where $V(r)$ is an arbitrary function equal to zero for $r \geq r_0$ ($r_0 < a$). We shall assume that along each χ-line v_φ varies as a linear function of the distance from the vertical axis $a + r\cos\chi$, i.e. $v_\varphi = U[b(r) - r\cos\chi]$ (U is a constant depending on the rotation rate of the fluid as a whole). If the magnetic field is sufficiently weak, then its braking influence on the plasma may be neglected and it may be assumed

motion as stationary (the kinematic problem). For large magnetic Reynolds numbers, during some time from the onset of motion in the cell the magnetic field obeys the induction law for an infinitely conducting fluid, $\partial\mathbf{H}/\partial t = \operatorname{rot}[\mathbf{vH}]$, or, in components,

$$\frac{\partial H_r}{\partial t} = -\frac{1}{r(a + r\cos\chi)}\left\{ r v_\varphi(r,\chi)\frac{\partial H_r}{\partial\varphi} + aV(r)\frac{\partial H_r}{\partial\chi}\right\}; \tag{1}$$

$$\frac{\partial H_\varphi}{\partial t} = \frac{1}{r}\left\{\frac{\partial}{\partial\chi}\left[v_\varphi(r,\chi)H_\chi - \frac{aV(r)}{a + r\cos\chi}H_\varphi\right] + \frac{\partial}{\partial r}\left[r v_\varphi(r,\chi)H_r\right]\right\}; \tag{2}$$

$$\frac{\partial H_\chi}{\partial t} = \frac{1}{a + r\cos\chi}\left\{a\frac{\partial}{\partial r}\left[V(r)H_r\right] + \frac{aV(r)}{a + r\cos\chi}\frac{\partial H_\varphi}{\partial\varphi} - v_\varphi(r,\chi)\frac{\partial H_\chi}{\partial\varphi}\right\}. \tag{3}$$

We shall seek the solution of equation (1) in the form $H_r = f(r,\chi)e^{i(\omega t + m\varphi)}$. Having determined the function $f(r,\chi)$ and applied to it the periodicity condition in χ, we obtain the discrete frequency spectrum

$$\omega_{mn}(r) = n\Omega(r) - m\Psi(r). \tag{4}$$

Here $\Omega(r) = V(r)/r$ (the frequency of motion of a fluid particle in χ), $\Psi(r) = Ub(r)/a$ (the mean velocity of its displacement in φ); $m, n = 0, \pm 1, \pm 2, \ldots$. Finally,

$$H_r^{mn} = A_r^{mn}(r) f_{mn}(r,\chi) e^{i[\omega_{mn}(r)t + m\varphi]}, \tag{5}$$

where

$$f_{mn} = \exp\{-in[\chi + (r/a)\sin\chi] + imU(r/a\Omega)(b/a+1)\sin\chi\},$$

and $A_r^{mn}(r)$ is an arbitrary function.

Taking into account that $\operatorname{div}\mathbf{H} = 0$, we eliminate H_φ from (3) and obtain a linear equation for H_χ with a right-hand side proportional to H_r. Substituting (5), one can find particular solutions H_χ^{mn}. But if one assumes that H_χ^{mn} depends on t as $e^{i\omega_{mn}(r)t}$, then it will represent a wave traveling in χ and φ, whose amplitude grows without bound with χ. A solution periodic in χ must contain time growth instead of spatial growth and must have the form

$$H_\chi^{mn} = \left[\tilde{A}_\chi^{mn}(r,\chi) + \tilde{B}_\chi^{mn}(r,\chi)t\right] f_{mn} e^{i[\omega_{mn}(r)t + m\varphi]}. \tag{6}$$

Then for $\tilde{B}_\chi^{mn}$ and $\tilde{A}_\chi^{mn}$ differential equations are obtained, one of which gives

$$\tilde{B}_\chi^{mn} = \frac{B_\chi^{mn}(r)}{a + r\cos\chi}$$

(B_χ^{mn} is an arbitrary function). From the second equation, after substituting $\tilde{B}_\chi^{mn}$ into it, $\tilde{A}_\chi^{mn}$ is found, depending on $\tilde{B}_\chi^{mn}(r)$ and the arbitrary function $A_\chi^{mn}(r)$. Applying to $\tilde{A}_\chi^{mn}(r,\chi)$ the periodicity requirement, we determine $B_\chi^{mn}(r)$. As a result,

$$H_\chi^{mn} = \frac{1}{a + r\cos\chi}\left[A_\chi^{mn}(r) - r\sin\chi\, A_r^{mn}(r) + ar\frac{d\Omega}{dr}A_r^{mn}(r)t\right] f_{mn} e^{i(\omega_{mn}t + m\varphi)}. \tag{7}$$

Finally, from equation (2) (it is also convenient to transform it using the equality $\operatorname{div}\mathbf{H} = 0$), after substitution of (5) and (7), quite analogously, in the form (6), H_φ^{mn} is found. The expression for A_φ^{mn} is cumbersome. Here it makes sense to write out only $\tilde{B}_\varphi^{mn}$, which determines the part of H_φ^{mn} growing in time:

$$\tilde{B}_\varphi^{mn} = U\left\{\frac{db}{dr}\left(1 + \frac{r}{a}\cos\chi\right) - \frac{1}{\Omega}\frac{d\Omega}{dr}[a + b(r)]\frac{r}{a}\cos\chi\right\} A_r^{mn}(r). \tag{8}$$

The system of functions $e^{-in[\chi + (r/a)\sin\chi]}$ is orthogonal with respect to the weight $1 + (r/a)\cos\chi$ and is complete, since it reduces to the system e^{-inz} by a one-to-one relation between z and χ. Any function of r, φ, χ can be represented by a collection $f_{mn}e^{im\varphi}$. To do this, it must be expanded in $e^{im\varphi}$, the expansion coefficients multiplied by $\exp\{-imU(r/a\Omega)(b/a+1)\sin\chi\}$, and expanded in e^{-inz}. Thus, A_r^{mn}, A_φ^{mn}, and A_χ^{mn}, in principle, can always be found from the initial conditions.

The factor $f_{mn}(r,\chi)e^{i(\omega_{mn}t+m\varphi)}$ describes the transport of the magnetic field in φ and χ. Because of the inhomogeneity of the flow, the spatial distribution of all harmonics $\mathbf{H}^{mn}$ with $m,n\neq 0$ becomes more complicated with time, since the periods of the functions $e^{i\omega_{mn}(r)t}$ in r become shorter. This inhomogeneity leads to stretching of the lines of force and to growth of the amplitudes H_φ^{mn} and H_χ^{mn}. The rate of growth depends on the gradients of the mean angular velocities Ω and Ψ. Terms proportional to t can, after a sufficiently long time, become dominant over all the others, including over H_r^{mn}. The growing part of the harmonic $\mathbf{H}^{00}$ is characterized by the fact that its distribution in space does not change. If $v_\varphi \ll v_\chi$, then the lower harmonics, nonzero in m with $n=0$, also preserve their regularity in space, slowly shifting in φ.

As Tverskoi showed ([1]), it is precisely the regular harmonics that are essential in calculating the large-scale field generated by the convective zone. Therefore, in studying the operation of a dynamo it is sufficient to isolate, in the solution found, the terms of the harmonics with zero n that grow with time. If the initial field within a cell is homogeneous and horizontal, then only the components with $m=1$ are excited, and $A_1^{10}(r)$ is expressed in terms of a Bessel function.

The solution scheme described is also suitable for a number of other problems. Let, in an orthogonal system of curvilinear coordinates q_1,q_2,q_3 with Lamé parameters h_1,h_2,h_3, the velocity vector of the liquid have the form $\mathbf{v}=\{v_1,v_2,0\}$, and let all streamlines lie on the surfaces $q_3=$ const. Introduce the "normalized" components of the vectors $\mathbf{H}$ and $\mathbf{v}$: $B_i=H_i/h_i$, $u_i=v_i/h_i$ ($i=1,2,3$; $u_3\equiv 0$), and denote $h_1h_2h_3=h$. Suppose, moreover, that u_1 and u_2 do not depend on q_1 (or on q_2; for definiteness we assume the former) and that the functions

$$\Phi(q_2,q_3)=\int_0^{q_2} u_2^{-1}\,dq_2,\qquad X(q_2,q_3)=\int_0^{q_2}(u_1/u_2)\,dq_2,$$

are defined everywhere in the flow region, and that u_2 nowhere vanishes. The equations $\operatorname{div}\mathbf{H}=0$ and $\operatorname{div}\mathbf{v}=0$ allow the induction law to be written as in Cartesian coordinates:

$$\partial B_1/\partial t = B_2\partial u_1/\partial q_2 + B_3\partial u_1/\partial q_3 - u_1\partial B_1/\partial q_1 - u_2\partial B_1/\partial q_2; \qquad (9)$$

$$\partial B_2/\partial t = B_2\partial u_2/\partial q_2 + B_3\partial u_2/\partial q_3 - u_1\partial B_2/\partial q_1 - u_2\partial B_2/\partial q_2; \qquad (10)$$

$$\partial B_3/\partial t = -u_1\partial B_3/\partial q_1 - u_2\partial B_3/\partial q_2. \qquad (11)$$

In a plane one can always choose an orthogonal coordinate net q_2,q_3 that includes a prescribed family of smooth vector lines of the vector $\mathbf{v}_2=\{0,v_2,0\}$. Therefore, when the flow is axisymmetric (q_1 is the azimuthal angle), or when the q_1-lines are parallel straight lines and the velocity field is symmetric with respect to transfer along these straight lines, the requirements introduced are satisfied and the solution written below is applicable. This type includes, for example, the outflow of the solar wind: an isotropic motion along radial straight q_2-lines is superposed with rotation about the axis of symmetry with an angular velocity that decreases with distance from the center. It is possible that more complicated velocity fields also satisfy the listed conditions, for which it is useful to solve the problem of the magnetic field.

The system (9)–(11) is solved in the same way as (1)–(3). The particular solution, depending on arbitrary k and ω, is as follows:

$$B_1^{k\omega} = \left\{A_1^{k\omega} + A_2^{k\omega} u_1 + A_3^{k\omega}\left(\partial X/\partial q_3 - u_1 \partial\Phi/\partial q_3\right) + \left(C_1^{k\omega} + C_2^{k\omega} u_1\right)(t-\Phi)\right\}\varphi^{k\omega}; \tag{12}$$

$$B_2^{k\omega} = u_2\left\{A_2^{k\omega} - A_3^{k\omega}\partial\Phi/\partial q_3 + C_2^{k\omega}(t-\Phi)\right\}\varphi^{k\omega}; \tag{13}$$

$$B_3^{k\omega} = A_3^{k\omega}\varphi^{k\omega}. \tag{14}$$

Here all $A_i^{k\omega}$ and $C_i^{k\omega}$ are arbitrary functions of q_3, and $\varphi^{k\omega} = e^{i\omega(t-\Phi)+ik(q_1-X)}$. Depending on the formulation of the problem, the solution satisfying the initial condition can be represented either by series of the functions (12)–(14), or by integrals of them with respect to k and ω. For example, in the solar-wind problem the flow region is infinite, and ω has a continuous spectrum of values. In this case one should put $C_i^{k\omega} = 0$ (the Fourier integral can also represent a nonperiodic dependence on t). If, however, the q_2-lines are closed (as in the case of a vortex), then to a given k there corresponds a discrete set of values of ω; periodicity with respect to q_2 also dictates a definite choice of $C_i^{k\omega}$. Formulas (12)–(14) are most convenient to use when the q_1-lines are also closed. If q_1 and q_2 vary from 0 to 2π, then ω_{mn} are given by formula (4), where one should put

$$\Omega(q_3) = 2\pi/\Phi(2\pi, q_3), \qquad \Psi(q_3) = X(2\pi, q_3)/\Phi(2\pi, q_3). \tag{15}$$

Then

$$C_1^{mn} = A_3^{mn}\left[d\Psi/dq_3 - (\Psi/\Omega)\, d\Omega/dq_3\right], \qquad C_2^{mn} = A_3^{mn}\Omega^{-1} d\Omega/dq_3, \tag{16}$$

where

$$\varphi^{mn} = \exp\{i\omega_{mn}(t-\Phi) + im(q_1 - X)\}.$$

If B_{0i} are the initial values of the components of the vector $\mathbf{B}$, then $A_j^{mn}(q_3)$ are determined as the coefficients of the expansions

$$F_{jm} e^{im(X-\Psi\Phi)} = \sum_n A_j^{mn} e^{-in\Omega\Phi}$$

in the system of functions of the variable q_2 orthogonal with respect to the weight $\partial(\Omega\Phi)/\partial q_2$, where $F_{jm}(q_2, q_3)$ are the amplitudes in the expansions

$$B_{01} - B_{02}u_1 + [(u_1 - \Psi)(\Phi/\Omega)d\Omega/dq_3 + \Phi\, d\Psi/dq_3 +$$

$$+u_1 \partial\Phi/\partial q_3 - \partial X/\partial q_3] B_{03} = \sum_m F_{1m} e^{imq_1},$$

$$B_{02}/u_2 + [\partial\Phi/\partial q_3 + (\Phi/\Omega)d\Omega/dq_3] B_{03} = \sum_m F_{2m} e^{imq_1},$$

$$B_{03} = \sum_m F_{3m} e^{imq_1}.$$

The main features of the solutions of the generalized problem are grasped by analogy with the case of a toroidal vortex. If $v_1 = 0$, then (12)–(14) pass into the formulas of paper [2], which are also valid when $\partial u_2/\partial q_1 \neq 0$.

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References

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[2] A. V. Getling, B. A. Tverskoi, *Astron. Zh.*, **45**, No. 3, 606 (1968).

Note: Figure translations are in progress. See original paper for figures.