ON THE NATURE OF THE SOLUTIONS OF LINEAR EVOLUTIONARY SYSTEMS WITH ELLIPTIC SPATIAL PART

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Abstract

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ON THE NATURE OF THE SOLUTIONS OF LINEAR EVOLUTIONARY SYSTEMS WITH ELLIPTIC SPATIAL PART

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In the paper ([1]) we studied positive (bounded below) solutions of some evolutionary hypoelliptic equations. It subsequently became clear that one can avoid the apparatus of fundamental solutions and a priori estimates (if one restricts oneself to L_1-theorems); this made it possible to include in the discussion some evolutionary non-hypoelliptic equations (for example, equations of the type of the equation of transverse vibrations of an elastic rod) and to weaken the restrictions on the coefficients. In addition, the results of ([1]) (in a strengthened version) can be extended from (positive bounded below) classical solutions to complex-valued generalized solutions belonging to certain cones. The latter makes it possible to pass to the study of solutions of systems of partial differential equations of arbitrary period. All this is set forth in the present note.

1. Lemmas on lateral smoothing

Consider the system

$$\mathcal{L}(t, x, \partial/\partial t, D_x)u = f(t, x). \tag{1}$$

Let the operator $\mathcal{L}(t, x, \partial/\partial t, D_x)$ satisfy the conditions:

1) The matrix $\mathcal{L}(t, x, 0, i\sigma)$ of size $N \times N$ is uniformly elliptic in the sense of I. G. Petrovskii; its highest order is r.
2) There exists $p > 1$ such that the elements $\mathcal{L}(t, x, \lambda, i\sigma)$, $l_{ij}(t, x, \lambda, i\sigma)$, contain terms of degrees $k_0, k_1, \dots, k_n$ in $\lambda, \sigma_1, \dots, \sigma_n$, respectively, for which

$$k_0 p + k_1 + \dots + k_n \leq r.$$

3) There exists a Lagrange-adjoint operator $\mathcal{L}^*(t, x, \partial/\partial t, D_x)$ with continuous bounded coefficients.

Let K^+ be a cone in the complex space C^N, the closure of which has only one common point with a certain half-space. Let $u(t,x)$ be a solution of system (1). Define the vector-functions

$$u^+(t,x) = \begin{cases} u(t,x), & \text{if } u \in K^+, \\ 0, & \text{if } u \notin K^+; \end{cases}$$

$$u^-(t,x) = \begin{cases} 0, & \text{if } u \in K^+, \\ u(t,x), & \text{if } u \notin K^+. \end{cases}$$

Then $u = u^+ + u^-$.

Denote by $\Pi^{a,x^0}_{(T_1,T_2)}$ the parallelepiped $T_1 < t < T_2$, $|x_j - x_j^0| < a$, $j = 1, \dots, n$, and by $\Sigma^{x^0}_a$ the cube $|x_j - x_j^0| < a$; $\Pi^a_{(T_1,T_2)} \equiv \Pi^{a,0}_{(T_1,T_2)}$, $\Sigma^0_a \equiv \Sigma_a$.

Lemma 1 (on interior lateral smoothing). Let:

1) $u(t,x)$ be a weak solution in $\Pi^2_{(t_1,t_2)}$ of the system $\mathcal{L}u \equiv f$;

2)

$$\iint_{\Pi^2_{(t_1,t_2)}} |f|\,dt\,dx = |f| < \infty;$$

3)

$$\iint_{\Pi^2_{(t_1,t_2)}} |u^-|\,dt\,dx < M;$$

4)

$$\iint_{\Pi^1_{(t_1,t_2)}} |u^+|\,dt\,dx < 1.$$

Then there exist positive constants $0 < a_1,\ h_1 > t_1,\ h_2 < t_2,\ \lambda_1$, depending only on the constant of uniform ellipticity δ and the constant A majorizing the moduli of the coefficients of the operator $\mathcal{L}^*$, such that

$$\iint_{\Pi^{1+a_1}_{(h_1,h_2)}} |u(t,x)|\,dt\,dx \le \lambda_1(1 + M + |f|). \tag{1'}$$

Lemma 2. Suppose conditions 1)–4) of Lemma 1 are satisfied ($t_1 = 0,\ t_2 = T$) and

5) $$\sum_{k=0}^{\max k_0-1} \int_{\Sigma_2} \left| \frac{\partial^k u(0,x)}{\partial t^k} \right| dx < B \quad \left(\text{or} \quad \sum_{k=0}^{\max k_0-1} \int_{\Sigma_2} \left| \frac{\partial^k u(T,x)}{\partial t^k} \right| dx < B \right).$$

Then there exist positive constants $a_2,\ h_2 < T,\ \lambda_2 > 1$, depending only on δ, A, such that

$$\iint_{\Pi^{1+a_2}_{(0,T-h_2)}} |u(t,x)|\, dt\, dx \leq \lambda_2(1 + M + |f| + B) \tag{2}$$

or

$$\iint_{\Pi^{1+a_2}_{(h_2,T)}} |u(t,x)|\, dt\, dx \leq \lambda_2(1 + M + |f| + B). \tag{2'}$$

Lemma 3 (metric). Suppose conditions 1)–4) of Lemma 1 are satisfied. Then for any $h \in (0, h_1 - t_1)$ (h_1 from Lemma 1) there exist positive constants a_3, λ_3, depending only on δ and A, such that

$$\iint_{\Pi^{1+a_3 h^{1/p}}_{(t_1+h,t_2-h)}} |u(t,x)|\, dt\, dx \leq \lambda_3(1 + M + |f| h^{r/p}). \tag{3}$$

Lemma 4 (metric). Suppose conditions 1)–5) of Lemmas 1, 2 are satisfied. Then for any $h \in (0, h_2)$ (h_2 from Lemma 2) there exist positive constants λ_4, a_4 (depending only on δ and A) such that

$$\iint_{\Pi^{1+a_4 h^{1/p}}_{(0,T-h)}} |u(t,x)|\, dt\, dx \leq \lambda_4(1 + Bh + M + |f| h^{r/p}) \tag{4}$$

or

$$\iint_{\Pi^{1+a_4 h^{1/p}}_{(h,T)}} |u(t,x)|\, dt\, dx \leq \lambda_4(1 + Bh + M + |f| h^{r/p}). \tag{4'}$$

2. Theorems on the growth of solutions in a layer. Uniqueness of the solution of the Cauchy problem

A simple consequence of Lemma 4 is the theorem describing the character of the growth of solutions of system (1) defined in the layer $\Pi^{\infty}_{(0,T)} \equiv \Pi_T$.

Theorem 1. Suppose:

1) $u(t,x)$ is a weak solution of the system $\mathcal{L}u = f$, continuous in Π_T;

2)

$$\iint_{\Pi^{2,x}_{(0,T)}} |f|\, dt\, dx \le F_1(|x|);$$

3)

$$\sum_{k=0}^{\max k_0 - 1} \int_{\Sigma_2^x} \left| \frac{\partial^k u(0,x)}{\partial t^k} \right| dx \le F_2(|x|)$$

$$\left(\text{or} \quad \sum_{k=0}^{\max k_0 - 1} \int_{\Sigma_2^x} \left| \frac{\partial^k u(T,x)}{\partial t^k} \right| dx \le F_2(|x|)\right);$$

4)

$$\iint_{\Pi^{2,x}_{(0,T)}} |u^-(t,x)|\, dt\, dx \leqslant F_3(|x|),$$

where $F_i(r)$, $i = 1, 2, 3$, are positive nondecreasing functions defined for $r \in [0, \infty)$.

Then

$$\iint_{\Pi^{2,x}_{(0,t)}} |u(t,x)|\, dt\, dx \leqslant C_1 \exp\left[C_2 \left(\frac{|x|}{(T-t)^{1/p}} \right)^{p/(p-1)} \right] \left(\sum_{j=1}^{3} F_j(|x|+2) + 1 \right) \tag{5}$$

(or

$$\iint_{\Pi^{2,x}_{(t,T)}} |u(t,x)|\, dt\, dx \leqslant C_1 \exp\left[C_2 \left(\frac{|x|}{t^{1/p}} \right)^{p/(p-1)} \right] \left(\sum_{j=1}^{3} F_j(|x|+2) + 1 \right). \tag{5'}$$

In particular, if $u(0,x) \equiv 0$, $f(t,x) \equiv 0$, $F_3(r) = \exp[rh(r)]$, where $h(r)$ is a positive nondecreasing function,

$$\iint_{\Pi^{2,x}_{(0,t)}} |u(t,x)|\, dt\, dx \leqslant C \exp\left[C_1 \left(\frac{|x|}{(T-t)^{1/p}} \right)^{p/(p-1)} \right] \left(\exp[(|x|+2)h(|x|+2)] + 1 \right). \tag{6}$$

From the last estimate, in the case of Petrovsky-parabolic systems, using the uniqueness theorem of G. N. Zolotarev ([2,3]), one obtains, in particular, the following new uniqueness theorem for the solution of the Cauchy problem.

Theorem 2. Let $u(t,x)$ be a solution in Π_T of a system, uniformly parabolic in the sense of Petrovsky, with Hölder-continuous bounded coefficients,

$$\mathcal{L}u = 0,$$

satisfying zero initial data, for which

$$|u^-(t,x)| \leqslant K\exp[|x|h(|x|)], \qquad \int^{\infty} \frac{dr}{h(r)^{p-1}} = \infty.$$

Then $u(t,x) \equiv 0$.

3. Growth of a solution defined in the whole space

As a simple consequence of Lemma 1, we give the following theorem.

Theorem 3. Let $u(t,x)$ be a solution in the whole space $(t, x_1, \dots, x_n)$ of the system

$$\mathcal{L}u = 0,$$

for which

$$\iint_{\Pi^2_{(-T_1,T_2)}} |u^-|\,dt\,dx \leqslant M(T_1,T_2)$$

for any positive T_1 and T_2. Then for any positive ε, T_1, T_2,

$$|u(t,x)| \leqslant C(\varepsilon, T_1, T_2)\exp[\varepsilon|x|^{p/(p-1)}] \tag{7}$$

in $\Pi_{(-T_1,T_2)}$.

Let us note that the equation

$$\frac{\partial u}{\partial t} = (-1)^{p-1}\frac{\partial^p u}{\partial x^p}$$

has a positive solution in the whole plane,

$$u_\varphi = \sum_{m=0}^{\infty} \exp\left[(-1)^{p-1}m^p t + mx - m^p\varphi(m)\right], \tag{8}$$

with

$$\lim_{m\to\infty}\varphi(m) = \infty,$$

which (because of $\varphi(m)$) does not satisfy the inequality

$$|u_\varphi| \leqslant K\exp[\psi(|x|)|x|^{p/(p-1)}], \qquad \lim_{r\to\infty}\psi(r) = 0,$$

where $\psi(r)$ tends to zero arbitrarily slowly.

Lemma on enclosure from below (for $p = 2k+1$) and from above (for $p = 2k$). Growth of solutions in the half-infinite (in t) cylinder. Let system (1) consist of a single equation with real coefficients, and let K^+ be a positive ray.

Lemma 5 (on enclosure from below for $p = 2k+1$, from above for $p = 2k$). *Let:*

1) $u(t,x)$ be a weak solution of the equation $\mathcal{L}u = f$ in $\Pi^2_{(-T,0)}$;

2)
$$\iint_{\Pi^2_{(-T,0)}} |f(t,x)|\,dt\,dx = |f| < \infty;$$

3)
$$\sum_{k=0}^{\max k_0 - 1} \int_{\Sigma_2} \left|\frac{\partial^k u(0,x)}{\partial t^k}\right|^2 dx < B \quad \text{for } p = 2k+1,$$
$$\sum_{k=0}^{\max k_0 - 1} \int_{\Sigma_2} \left|\frac{\partial^k u(-T,x)}{\partial t^k}\right|^2 dx < B \quad \text{for } p = 2k;$$

4)
$$\iint_{\Pi^2_{(-T,0)}} |u^-(t,x)|\,dt\,dx < M;$$

5)
$$\iint_{\Pi^1_{(-T_1,0)}} |u^+|\,dx\,dt < 1 \quad (p = 2k{+}1); \qquad \iint_{\Pi^1_{(-T,-T_1)}} |u^+|\,dx\,dt < 1 \quad (p = 2k).$$

Then there exist positive constants a_5, λ_5, and $h_3 < T - T_1$, depending only on δ and A, such that for $p = 2k+1$

$$\iint_{\Pi^{1+a_5}_{(-T_1-h_3,0)}} |u(t,x)|\,dt\,dx \le \lambda_5(1 + B + M + |f|),$$

and for $p = 2k$

$$\iint_{\Pi^{1+a_5}_{(-T,-T_1+h_3)}} |u(t,x)|\,dt\,dx \le \lambda_5(1 + B + M + |f|). \tag{9}$$

A simple consequence of Lemma 4 is

Theorem 4. *Let:*

1) $u(t,x)$ be a solution of the hypoelliptic equation $\mathcal{L}u = 0$ in $\Pi_{(-\infty,0]}$ for $p = 2k+1$; for $p = 2k$, $u(t,x)$ be a solution of the equation $\mathcal{L}u = 0$ in $\Pi_{[0,\infty)}$;

2)
$$\iint_{\Pi^a_{(-\infty,0]}} |u^-(t,x)|\,dt\,dx < M \quad (p = 2k+1),$$

$$\iint_{\Pi^a_{[0,\infty)}} |u^-(t,x)|\,dt\,dx < M \quad \text{for } p = 2k.$$

Then in every $\Pi^\eta_{(-\infty,0]}$ *for* $p = 2k+1$ *and in every* $\Pi^\eta_{[0,\infty)}$ *for* $p = 2k$, $0 < \eta < a$,

$$|u(t,x)| \leq C_1(\eta) \exp[C_2(\eta)t]. \tag{10}$$

Example (8) shows that for $p = 2k+1$ inequality (10) is not valid in $\Pi^\eta_{[0,\infty)}$, and for $p = 2k$ in $\Pi^\eta_{(-\infty,0]}$.

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Note: Figure translations are in progress. See original paper for figures.