SOME ANALYTIC QUESTIONS AND THE INVERSE STURM-LIOUVILLE PROBLEM FOR AN EQUATION WITH A HIGHLY SINGULAR POTENTIAL

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SOME ANALYTIC QUESTIONS AND THE INVERSE STURM-LIOUVILLE PROBLEM FOR AN EQUATION WITH A HIGHLY SINGULAR POTENTIAL

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1. As is known, the question of reconstructing the equation

$$y'' + [\lambda - V(x)]y = 0, \qquad 0 \le x < \infty, \tag{1}$$

with continuous real potential $V(x)$ from the spectral function (s.f.) has been completely solved and investigated in papers ([1–3]). The analogous problem, when the potential has at $x = 0$ a Bessel singularity $n(n+1)x^{-2}$, was subsequently studied in ([4]). We shall consider the case where $V(x)$ has at $x = 0$ a singularity of higher order: $V(x) = W(x) + U(x)$, where $W(x) \to +\infty$ as $x \to 0$,

$$W'(x) = O(W^{3/2-\delta}(x)), \qquad 0 < \delta < \tfrac{1}{2}, \tag{2}$$

$W(x) > 0$ $(0 < x < 1)$; $W(x) \equiv 0$ $(x \ge 1)$, and satisfies all the remaining conditions from ([5]) (sufficient smoothness, multiple monotonicity ([6]), etc.). The addition $U(x) \in L(\varepsilon, a)$ for any $0 < \varepsilon < a < \infty$, $U(x)W^{-1/2}(x) \in L(0, \frac{1}{2})$. Additional restrictions on $U(x)$ are introduced below as needed. The regular solution $\varphi(x,k)$ of equation (1) with $\lambda = k^2$ is defined ([7,5]) by the condition, as $x \to 0$:

$$\varphi(x,k)/\gamma_0(x) \to 1; \qquad \gamma_0(x) = 2^{-1/2}W^{-1/4}(x)\exp\left\{-\int_x^1 W^{1/2}(t)\,dt\right\}. \tag{3}$$

Theorem 1. *Under the listed conditions there exists a nondecreasing (possibly nonunique) s.f. $\rho(\lambda)$, generating Parseval' s equality and the inversion formula for equation (1):*

$$F(\lambda) = \int_0^\infty f(x)\varphi(x,\sqrt{\lambda})\,dx, \qquad f(x) = \int_{-\infty}^\infty F(\lambda)\varphi(x,\sqrt{\lambda})\,d\rho(\lambda), \tag{4}$$

where $f(x) \in L^2(0,\infty)$, and the integrals converge in $L^2_\rho(-\infty,\infty)$ and $L^2(0,\infty)$, respectively. Among the s.f. of problem (1), (3) there always exist orthogonal s.f. Any s.f. $\rho(\lambda)$ of problem (1), (3) is such that: 1) the density of the set of growth points of $\rho(k^2)$ on the half-axis $0 < k < \infty$ is infinite; 2) $\exp\{x\sqrt{|\lambda|}\} \in L\{(-\infty,0), d\rho(\lambda)\}$ for $0 < x < \infty$.

Lemma. Let $X = X(k)$ be the root of the equation $W(X) = k^2$,

$$\omega(k) = \left(\frac{k}{2}\right)^{1/2} \exp\left\{\int_{X(k)}^1 \sqrt{W(t)}\,dt + k^2\int_0^{X(k)} \frac{dt}{\sqrt{W(t)-k^2}+\sqrt{W(t)}}\right\}. \tag{5}$$

* The first part of the theorem can be proved by known methods ([8,9]); for a particular case see ([7]). Assertions 1), 2) are similar to those known for cases ([1,10,4]).

Then $\omega(k)$ is continuous, $\omega(0) = 0$, and if, for some $p > 2$, there exists

$$\lim_{x\to 0} \widetilde{W}(x)x^p = C > 0,$$

then, as $k \to +\infty$,

$$\ln\omega(k) \sim B_p k X(k), \qquad B_p = \frac{2}{p-2} + \frac{1}{p}\int_1^\infty \frac{t^{-1-1/p}\,dt}{\sqrt{t-1}+\sqrt{t}}. \tag{6}$$

Formulae (6) are also valid when, for every $\varepsilon > 0$, as $x \to 0$,

$$W(x)x^{p+\varepsilon} \to 0, \qquad W(x)x^{p-\varepsilon} \to \infty$$

monotonically in some neighborhood of $x = 0$.

Theorem 2. Let $U(x) = p(x) + r(x)$ and, for some $\varepsilon > 0$,

$$p(x)W^{-1/3}(x) \in L(0,\varepsilon), \qquad \sup_{0\le x\le\varepsilon} |r(x)|W^{-1/2}(x)x^{1-\varepsilon} < \infty. \tag{7}$$

Then, for any s.f. $\rho(\lambda)$ of problem (1), (3),

$$kW^{-1}(X)\omega^{-2}(k) \in L\{(1,\infty), d\rho(k^2)\},$$

and if $V(x) \in L(1,\infty)$, then

$$d\rho(k^2)/dk \sim 2\pi^{-1}\omega^2(k), \qquad k \to +\infty. \tag{8}$$

Formulae (8), (6) make it possible, from the asymptotics of the spectral density, to find the asymptotics of $W(x)$ as $x \to 0$.

Theorem 3. Under the conditions of Theorem 1, $\varphi(x,k)$, for each $x > 0$, is an even entire function of k of exponential type x,

$$|\varphi(x,k)| \le C(a)\exp\{x|\operatorname{Im} k|\}\min\{\gamma_0(x);1\}, \qquad 0 \le x \le a,$$

and, as $k \to i\infty$,

$$\ln\varphi(x,k) \sim x|k|$$

uniformly for $x \in [\varepsilon, a]$.

As $k \to +\infty$, under condition (7), uniformly for $x \in [\varepsilon, a]$ we have

$$\varphi(x,k) = \omega^{-1}(k)\{\sin[kx + \beta(k)] + o(1)\}^*,$$

and uniformly for $x \in [0,a]$ the asymptotic formula

$$\varphi(x,k) \simeq \frac{\pi}{2}\left(\frac{k\xi}{6}\right)^{1/2}\omega^{-1}(k)\big(k^2 - W(x)\big)^{-1/4}\{J_{1/3}(\xi) + J_{-1/3}(\xi)\},$$

is valid, where

$$\xi = \xi(x,k) = \int_{X(k)}^{x}\big(k^2 - W(t)\big)^{1/2}\,dt, \qquad \arg\xi = \begin{cases} 0, & x \ge X(k), \\ 3\pi/2, & x < X(k). \end{cases}$$

The indicated asymptotic formulae admit differentiation with respect to x.

2. As N. I. Akhiezer showed ([6], [11]), the Wiener–Paley theorem admits a generalization to the case of the Fourier–Bessel integral with functions J_ν of arbitrary index $\nu \ge -1/2$ (for half-integer ν, see ([4])). Under the conditions of Theorem 1 together with (7), the following analogue of the Wiener–Paley theorem is valid.

Theorem 4. In order that the function $G(k)$ admit the representation

$$G(k) = \int_0^\sigma g(t)\varphi(t,k)\,dt, \qquad \sigma < \infty, \tag{9}$$

where $g(t) \in L^2(0,\sigma)$, it is necessary and sufficient that $G(k)$ be an even entire function of exponential type $\le \sigma$ and that

$$G(k)\omega(k) \in L^2(0,\infty).$$

Theorem 4 makes it possible to establish the existence of transformation operators**:

Theorem 5. Let $\varphi_i(x,k)$ be the regular solution of equation (1) with potential

$$V_i(x) = W(x) + U_i(x), \qquad U_i(x) \quad (i = 1, 2)$$

satisfying (7) and, for some $\varepsilon > 0$,

$$x^{1/2-\varepsilon} U_{ij}(x) W^{-5/6}(x) \in L(0,\varepsilon) \qquad (i,j = 1,2), \tag{10}$$

* The expression $\alpha(k) = \beta(k) - \pi/4$ see ([5]). Under the conditions of the lemma, $\beta(k) \sim A_p k X(k)$.

** Transformation operators, well known since the works of A. Ya. Povzner, B. M. Levitan, and V. A. March

where $U_{ij}(x) = U_i(x) - U_j(x)$, and δ is the same as in (2). Then*

$$\varphi_i(x,k) = \varphi_j(x,k) + \int_0^x K_{ij}(x,t) \varphi_j(t,k)\, dt, \tag{11}$$

and, for any $0 < a < \infty$,

$$\sup_{0<x<a} \int_0^x K_{ij}^2(x,t)\, dt < \infty.$$

With the aid of the operators (11), as also in ([3]), one establishes

Theorem 6. Let $V_i(x)$ $(i = 1,2)$ be the same as in Theorem 5, and let $\rho_i(\lambda)$ be some s.f. corresponding to problem (1), (3) with $V(x) = V_i(x)$. Then, if $\rho_1(\lambda) = \rho_2(\lambda)$, then $V_1(x) = V_2(x)$ almost everywhere $(0 < x < \infty)$.

3. The differential properties of the kernels of the transformation operators are derived from (11) and (4) with the aid of the following theorem.

Theorem 7. In the notation of Theorem 5, let $U_i(x)$ $(i = 1,2)$ satisfy (7), be continuous together with $U_i'(x)$, $|U_i'(x)| \le C|W'(x)|$ for $0 < x \le 1/2$, $C > 0$; let $U_{12}(x)$ be twice differentiable and, for any $a < \infty$,

$$W^{2-\nu}(x) U_{12}^{(\nu)}(x) \in L(0,a) \qquad (\nu = 0,1,2).$$

Then

$$\varphi_2(x,k) = \varphi_1(x,k) + a_1(x)\varphi_1' k^{-2} + a_2(x)\varphi_1 k^{-2} + R(x,k),$$

where**

$$a_1(x) = \frac{1}{2}\int_0^x U_{12}(t)\, dt, \qquad a_2(x) = -\frac{1}{4} U_{12}(x) - \frac{1}{2} a_1^2(x),$$

and for $k \geq 1,\ 0 \leq x \leq a$,

$$\left|\frac{\partial^\nu R(x,k)}{\partial x^\nu}\right| \leq C(a)k^{\nu-2}G(k) \qquad (\nu = 0,1);$$

$$G(k)\omega(k) = k^{1/2}|W'(X)|^{-1/2} \in L^2(1,\infty).$$

Definition. $Q_W(\Omega)$ is the set of functions $g(x,y)$ in

$$\Omega = \{x,y \mid 0 \leq y \leq x,\ 0 \leq x < \infty\}$$

such that:

1) $g(x,y)$, $g'_x(x,y)$, $g'_y(x,y)$ are continuous in Ω;
2) if

$$N_a\{h(x,y)\} = \sup_{0<x<a} \|h(x,\cdot)\|_{L^2(0,x)},$$

then, for any $a < \infty$,

$$N_a\{g(x,y)W(y)\} < \infty, \qquad N_a\{g'_x(x,y)W^{1/2}(y)\} < \infty;$$

3) $g'_y(x,y)$ is absolutely continuous in y for each $x > 0$; $g'_x(x,y)$, as a vector-function of x in $L^2\{(0,b);dy\}$, for any $b > 0,\ x \geq b$, has a continuous strong derivative with respect to x, $g''_{xx}(x;(y))$, and

$$N_a\{g''_{xx}(x;(y))\} + N_a\{g''_{yy}(x,y)\} < \infty;$$

4) there exist and are continuous $d^\nu g(x,x)/dx^\nu$, $(\nu = 1,2)$, and

$$\{W^{3/2}(x)|dg(x,x)/dx| + W(x)|d^2g(x,x)/dx^2|\} \in L(0,1) \qquad (12)$$

***.

Theorem 8. Under the conditions of Theorem 7 the kernels of the transformation operators $K_{ij}(x,y) \in Q_W(\Omega)$,

$$2\,dK_{ij}(x,x)/dx = U_{ij}(x) \qquad (K_{ij}(0,0) = 0),$$

and for $0 < x < \infty$ and almost all $y \in (0,x)$,

$$K''_{ijxx}(x;(y)) - V_i(x)K_{ij}(x,y) = K''_{ijyy}(x,y) - V_j(y)K_{ij}(x,y). \qquad (13)$$

4. Guided by the scheme of I. M. Gelfand–B. M. Levitan ([1]) and relying on the results obtained, we arrive at the following theorem.

Theorem 9. Let $V_i(x) = W(x) + U_i(x)$ $(i = 1, 2)$ be the same as in Theorem 7; let $\rho_1(\lambda)$ be an s.f. of problem (1), (3) with $V_1(x)$. In order that a nondecreasing function $\rho(\lambda)$ be an s.f. of problem (1), (3) with $V_2(x)$, it is necessary:

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* Obviously, the kernels $K_{12}(x, t)$ and $K_{21}(x, t)$ generate mutually inverse operators.

** Under more stringent conditions the asymptotic expansion can be continued:

$$R(x, k) = a_3(x)\varphi_1' k^{-4} + a_4(x)\varphi_1 k^{-4} + \cdots$$

*** It follows from the definition that if $g(x, y) \in Q_W$, then $g(x, 0) = g_y'(x, 0) = 0$, and, by virtue of Theorem 3.4.2 from ([13]), almost everywhere in Ω there exists $g_{xx}''(x, y)$, equal to $g_{xx}''(x; (y))$.

1) property 1) of Theorem 1; 2) the existence of $f(x, y) \in Q_W(\Omega)$, where

$$f(x, y) = \frac{\partial^2}{\partial x\,\partial y} \int_{-\infty}^{\infty} \left\{ \int_0^x \varphi_1(t, \sqrt{\lambda})\, dt \right\} \left\{ \int_0^y \varphi_1(t, \sqrt{\lambda})\, dt \right\} d\{\rho(\lambda) - \rho_1(\lambda)\}. \tag{14}$$

Conditions 1), 2) are sufficient in order that the nondecreasing $\rho(\lambda)$ be the s.f. of problem (1), (3) with continuously differentiable potential

$$V(x) = V_1(x) + 2dK(x, x)/dx, \tag{15}$$

where $K(x, y) \in Q_W(\Omega)$ is determined uniquely by the solvable equation

$$K(x, y) + f(x, y) + \int_0^x K(x, t) f(t, y)\, dt = 0 \tag{16}$$

and is the kernel (11) that carries $\varphi_1(x, k)$ into $\varphi(x, k)$. $V(x) - V_1(x) = o(W^{-1}(x))$ as $x \to 0$ and satisfies the integral estimates following from (12).

Sufficient conditions alone for the solvability of the inverse problem of spectral analysis are formulated much more simply:

Theorem 10. Let $\rho_1(\lambda)$ be the s.f. of problem (1), (3) with $V_1(x)$, as in Theorem 1. If $\rho(\lambda)$ is nondecreasing, has property 1) of Theorem 1, and $f(x, y)$ (14) has in Ω continuous derivatives of order $n + 1$ $(n \geq 1)$, then $\rho(\lambda)$ is the s.f. of problem (1), (3), where $V(x)$ is determined from (15), (16), (14), and $V(x) - V_1(x)$ has, for $x \geq 0$, n continuous derivatives.

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