ON THE CLASSICAL SOLUTION OF THE MULTIDIMENSIONAL STEFAN PROBLEM

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ON THE CLASSICAL SOLUTION OF THE MULTIDIMENSIONAL STEFAN PROBLEM

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In the present note we prove the existence, uniqueness, and stability with respect to perturbations of the initial data of a regular classical solution of the multidimensional Stefan problem (problem A) (cf. ([1,2])). The exposition is given for the case when the number of spatial independent variables $N = 2$, but the method and the result are easily extended to the case $N > 2$.

Problem A. Suppose it is required to find the temperature $u = u(x_1, x_2, t)$ in $\overline{Q}_T : \overline{\Pi} \times [0, T]$, where $\Pi : \ \{(x_1, x_2),\ 0 = l_{11} \le x_1 \le l_{21} = l_1,\ 0 = l_{12} \le x_2 \le l_{22} = l_2\}$, and the surface $S :\ x_1 = x_1(s, t),\ x_2 = x_2(s, t),\ s^I \le s \le s^{\mathrm{II}},\ 0 \le t \le T$, of phase separation, representing the set of all points of $\overline{Q}_T$ at which $u = 0$, and dividing $\overline{Q}_T$ into two regions, in one of which $u < 0$, and in the other $u > 0$, under the following conditions.

At the points $Q_T = \Pi \times (0, T]$, where $\Pi :\ \{(x_1, x_2),\ l_{11} < x_1 < l_{21},\ l_{12} < x_2 < l_{22}\}$, not belonging to S, $u(x_1, x_2, t)$ satisfies the equation

$$u_{it} = a_i^2 (u_{ix_1x_1} + u_{ix_2x_2}), \qquad a_i = \mathrm{const} > 0, \qquad i = 1, 2, \tag{1}$$

where $i = 1$ refers to that part of $\overline{Q}_T$ where $u < 0$, $i = 2$ to that where $u > 0$; moreover on S the relations

$$u_1 = u_2 \equiv 0, \tag{2}$$

$$\Phi_t + (k_1 \,\mathrm{grad}\, u_1 - k_2 \,\mathrm{grad}\, u_2, \mathrm{grad}\, \Phi) \equiv 0, \quad k_1 = \mathrm{const}, \quad k_2 = \mathrm{const}, \tag{3}$$

hold, where $\Phi(x_1, x_2, t) = 0$ is the equation of the surface S, obtained by eliminating the parameter s from the equations $x_1 = x_1(s, t)$, $x_2 = x_2(s, t)$.

On $\partial_b \overline{Q}_T = \Gamma \times [0, T]$, where Γ is the boundary of Π, it must be that

$$\partial u/\partial n\big|_{\partial_b \overline{Q}_T} = F(x,t), \qquad x=(x_1,x_2)\in\Gamma, \qquad 0\le t\le T, \tag{4}$$

where $\partial/\partial n$ is differentiation along the normal to $\partial_b\overline{Q}_T$. For $t=0$ there must be

$$u\big|_{t=0} = \varphi(x_1,x_2), \qquad x\in\overline{\Pi}; \tag{5}$$

$$x_1(s,0)=\psi_1(s), \qquad x_2(s,0)=\psi_2(s), \qquad s^I\le s\le s^{\mathrm{II}}, \tag{6}$$

where $x_1=\psi_1(s)$, $x_2=\psi_2(s)$, $s^I\le s\le s^{\mathrm{II}}$, is a simple nonclosed smooth curve (in particular, $\psi_1'(s)$ and $\psi_2'(s)$ are continuous and $\psi_1'^2(s)+\psi_2'^2(s)\neq 0$ for $s^I\le s\le s^{\mathrm{II}}$), connecting an interior point of the edge $[l_{11}\le x_1\le l_{21},\ x_2=l_{12},\ t\equiv 0]$ with an interior point of the edge $[l_{11}\le x_1\le l_{21},\ x_2=l_{22},\ t\equiv 0]$ and dividing $\overline{\Pi}$ into two regions, in one of which $\varphi(x_1,x_2)<0$, and in the other $\varphi(x_1,x_2)>0$, moreover the interior points of this curve are interior points of the face $\overline{\Pi}$.

In the definition of a classical solution of problem A one is concerned with the surface $S:\ x_1=x_1(s,t)$, $x_2=x_2(s,t)$, $s^I\le s\le s^{\mathrm{II}}$, $0\le t\le T$, of phase separation, satisfying the following requirements:

Requirements B. 1) S is a simple nonclosed smooth surface (in particular, $x_{1t}(s,t)$, $x_{2t}(s,t)$, $x_{1s}(s,t)$, $x_{2s}(s,t)$ are continuous and

$x_1^2(s,t)+x_2^2(s,t)\neq 0$ for $s^I\le s\le s^{\mathrm{II}}$, all interior points of which are interior points of $\overline{Q}_T$. 2) The boundary S is a closed curve consisting of four pieces

$$\{\, l_{11}<x_1=x_1(s^I,t)<l_{21},\ x_2=x_2(s^I,t)=l_{12},\ 0\le t\le T\,\},$$

$$\{\, l_{11}<x_1=x_1(s^{\mathrm{II}},t)<l_{21},\ x_2=x_2(s^{\mathrm{II}},t)=l_{22},\ 0\le t\le T\,\},$$

$$\{\, l_{11}<x_1=x_1(s,0)=\psi_1(s)<l_{21}, \quad 0\le x_2=x_2(s,0)=\psi_2(s)\le l_2,\ s^I\le s\le s^{\mathrm{II}},\ \psi_2(s)\neq l_{12},l_{22}$$

$$\text{for } s\neq s^I,s^{\mathrm{II}}\,\},$$

$$\{\, l_{11}<x_1=x_1(s,T)<l_{21},\ l_{12}\le x_2=x_2(s,T)\le l_{22},\ s^I\le s\le s^{\mathrm{II}},$$

$$x_2(s,T)\neq l_{12},l_{22} \text{ for } s\neq s^I,s^{\mathrm{II}}\,\}.$$

Definition 1. By a classical solution of problem A (problem (1)–(6)) we shall mean functions $u = u(x_1, x_2, t)$ and a surface $S:\ x_1 = x_1(s,t),\ x_2 = x_2(s,t)$, $s^I \le s \le s^{\text{II}}$, $0 \le t \le T$, such that: 1) the surface S satisfies the requirements B; 2) the function $u = u(x_1, x_2, t)$ is continuous in $\overline{Q}_T$, and its derivatives u_{x_1}, u_{x_2}, $u_{x_1x_1}$, $u_{x_1x_2}$, $u_{x_2x_2}$, u_t are continuous in the closed regions into which the surface S divides the parallelepiped $\overline{Q}_T$; 3) all relations (1)–(6) are fulfilled.

Lemma. If the function $u = u(x_1, x_2, t)$ and the surface $S:\ x_1 = x_1(s,t)$, $x_2 = x_2(s,t)$ satisfy all the conditions of Definition 1 with (3) replaced by

$$\partial x_1/\partial t = k_1 \partial u_1/\partial x_1 - k_2 \partial u_2/\partial x_1,$$

$$\partial x_2/\partial t = k_1 \partial u_1/\partial x_2 - k_2 \partial u_2/\partial x_2 \quad \text{on } S, \tag{3'}$$

then $u = u(x_1, x_2, t)$ and $S:\ x_1 = x_1(s,t)$, $x_2 = x_2(s,t)$ form a classical solution of problem A.

Definition 2. A function $u = u(x_1, x_2, t)$ and a surface $S:\ x_1 = x_1(s,t)$, $x_2 = x_2(s,t)$, satisfying the conditions of the lemma, will be called a regular classical solution of problem A. To prove the existence, uniqueness, and stability with respect to perturbations of the initial data of a regular classical solution of problem A, we reduce it, replacing relation (3) by relations $(3')$, to an equivalent system of nonlinear Volterra integral equations of the second kind.

Let us first specify the formulation of condition (4) of problem A, writing it for each of the lateral faces of $\overline{Q}_T$:

$$u_{1x_1}(l_{11}, x_2, t) = F_1(x_2, t), \qquad u_{2x_1}(l_{21}, x_2, t) = F_2(x_2, t),$$

$$l_{12} \le x_2 \le l_{22}, \qquad 0 \le t \le T; \tag{4_1}$$

$$u_{ix_2}(x_1, l_{12}, t) = F_{i1}(x_1, t); \qquad i = 1,\ l_{11} \le x_1 \le x_1(s^I, t);$$

$$i = 2,\ x_1(s^I, t) \le x_1 \le l_{21}, \qquad 0 \le t \le T; \tag{4_2}$$

$$u_{ix_2}(x_1, l_{22}, t) = F_{i2}(x_1, t); \qquad i = 1,\ l_{11} \le x_1 \le x_1(s^{\text{II}}, t);$$

$$i = 2,\ x_1(s^{\text{II}}, t) \le x_1 \le l_{21}, \qquad 0 \le t \le T. \tag{4_3}$$

Next we introduce the "Green functions" :

$$\Pi_1(x_1, x_2, t; \xi_1, \xi_2, \tau) = G_{11}(x_1, t, \xi_1, \tau) G_{21}(x_2, t, \xi_2, \tau),$$

$$\Pi_2(x_1, x_2, t; \xi_1, \xi_2, \tau) = G_{12}(x_1, t, \xi_1, \tau) G_{22}(x_2, t, \xi_2, \tau),$$

where G_{11} and G_{12} are Green functions for the half-lines $0 \le k_1 < +\infty$ and $-\infty < x_1 \le l_1$ with coefficients a_1 and a_2, while G_{2i}, $i = 1, 2$, are Green functions for the segment $0 \le x_2 \le l_2$ with coefficients a_i, $i = 1, 2$ ([3]).

For a classical solution of problem A the representation holds

$$u_1(x_1, x_2, t) = \iint_{\tau=0} \varphi(\xi_1, \xi_2)\, \Pi_1(x_1, x_2, t, \xi_1, \xi_2, 0)\, d\xi_1\, d\xi_2 +$$

$$+ a_1^2 \iint_{\xi_1=0} F_1(\xi_2, \tau)\, \Pi_1(x_1, x_2, t, 0, \xi_2, \tau)\, d\xi_2\, d\tau +$$

$$+ a_1^2 \iint_{\xi_2=0} F_{11}(\xi_1, \tau)\, \Pi_1(x_1, x_2, t, \xi_1, 0, \tau)\, d\xi_1\, d\tau +$$

$$+ a_1^2 \iint_{\xi_2=l_2} F_{12}(\xi_1, \tau)\, \Pi_1(x_1, x_2, t, \xi_1, l_2, \tau)\, d\xi_1\, d\tau +$$

$$+ a_1^2 \iint_{\xi_1=x_1(s,\tau),\, \xi_2=x_2(s,\tau)} u_{1\xi_1}(x_1(s,\tau), x_2(s,\tau), \tau)\, \Pi_1(x_1, x_2, t, x_1(s,\tau), x_2(s,\tau), \tau)\, d\xi_2\, d\tau +$$

$$+ a_1^2 \iint_{\xi_1=x_1(s,\tau),\, \xi_2=x_2(s,\tau)} u_{1\xi_2}(x_1(s,\tau), x_2(s,\tau), \tau)\, \Pi_1(x_1, x_2, t, x_1(s,\tau), x_2(s,\tau), \tau)\, d\xi_1\, d\tau, \quad (7_1)$$

$$u_2(x_1, x_2, t) = \iint_{\tau=0} \varphi(\xi_1, \xi_2)\, \Pi_2(x_1, x_2, t, \xi_1, \xi_2, 0)\, d\xi_1\, d\xi_2 +$$

$$+ a_2^2 \iint_{\xi_1=l_1} F_2(\xi_2, \tau)\, \Pi_2(x_1, x_2, t, l_1, \xi_2, \tau)\, d\xi_1\, d\tau +$$

$$+ a_2^2 \iint_{\xi_2=0} F_{21}(\xi_1, \tau)\, \Pi_2(x_1, x_2, t, \xi_1, 0, \tau)\, d\xi_1\, d\tau +$$

$$+ a_2^2 \iint_{\xi_2=l_2} F_{22}(\xi_1, \tau)\, \Pi_2(x_1, x_2, t, \xi_1, l_2, \tau)\, d\xi_1\, d\tau +$$

$$+a_2^2 \iint_{\xi_1=x_1(s,\tau),\,\xi_2=x_2(s,\tau)} u_{2\xi_1}(x_1(s,\tau),x_2(s,\tau),\tau)\,\Pi_2(x_1,x_2,t,x_1(s,\tau),$$

$$x_2(s,\tau),\tau)\,d\xi_2\,d\tau + a_2^2 \iint_{\xi_1=x_1(s,\tau),\,\xi_2=x_2(s,\tau)} u_{2\xi_2}(x_1(s,\tau),x_2(s,\tau),\tau)\times$$

$$\times\Pi_2(x_1,x_2,t,x_1(s,\tau),x_2(s,\tau),\tau)\,d\xi_1\,d\tau. \qquad (7_2)$$

Introduce the notation:

$$u_{ix_j}(x,(s,t),x_2(s,t),t) = v_{ij}(s,t),$$

$$u_{ix_jx}k(x_1(s,t),x_2(s,t),t) = p_{ijk} = u_{ix_kx}j(x_1(s,t),x_2(s,t),t) = p_{ikj}, \qquad (8)$$

$$x_{is}(s,t) = z_i(s,t), \qquad i=1,2;\ j=1,2;\ k=1,2;$$
$$s^I \le s \le s^{\mathrm{II}}, \quad 0 \le t \le T.$$

Theorem 1. Let, for the initial data of problem A, the following conditions be satisfied:

a) $F_i(x_2,t)$, $F_{ix_2}(x_2,t)$, $F_{it}(x_2,t)$, $i=1,2$, are continuous for $0=l_{12}\le x_2 \le l_{22}=l_2,\ 0\le t\le T$;

b) $F_{ij}(x_1,t),\ F_{ijx_1}(x_1,t),\ F_{ijt}(x_1,t),\ i=1,2;\ j=1,2,$ are continuous for $0=l_{11}\le x_1\le l_{21},\ 0\le t\le T$;

c) $\psi_1(s),\ \psi_2(s),\ \psi_1'(s),\ \psi_2'(s)$ are continuous, $\psi_1'^2(s)+\psi_2'^2(s)\ne 0$ for $s^I \le s\le s^{\mathrm{II}}$;

d) $\varphi(x_1,x_2)$ is continuous for $0=l_{11}\le x_1\le l_{21},\ 0=l_{12}\le x_2\le l_{22}$; $\varphi,\ \varphi_x,\ \varphi_{x_ix_j}$ are continuous in both closed domains into which the curve $x_1=\psi_1(s),\ x_2=\psi_2(s),\ s^I\le s\le s^{\mathrm{II}}$, divides the face $\Pi:\ 0\le x_1\le l_1,\ 0\le x_2\le l_2$;

e) the compatibility conditions are satisfied.

Then problem A with relations $(3')$ instead of (3) is equivalent, by virtue of (7_i), (8), to the system of nonlinear integral equations B for $x_{1s}x_{2s}\ne 0$.

System B.

$$v_{ij}(s,t) = \iint_{\tau=0} \varphi(\xi_1,\xi_2)\,\Pi_{ix_j}\,d\xi_1\,d\xi_2 + a_i^2 \iint_{\xi_1=l_{i1}} F_i(\xi_2,\tau)\,\Pi_{ix_j}\,d\xi_2\,d\tau +$$

$$+a_i^2 \iint_{\xi_2=0} F_{i1}(\xi_1,\tau)\,\Pi_{ix_j}\,d\xi_1\,d\tau + a_i^2 \iint_{\xi_2=l_2} F_{i2}(\xi_1,\tau)\,\Pi_{ix_j}\,d\xi_1\,d\tau +$$

$$+\,a_i^2 \iint_{\xi_1=x_1(s,\tau),\,\xi_2=x_2(s,\tau)} v_{i1}\,\Pi_{ix_j}\,d\xi_2\,d\tau + a_i^2 \iint_{\xi_1=x_1(s,\tau),\,\xi_2=x_2(s,\tau)} v_{i2}(s,\tau)\,\Pi_{ix_j}\,d\xi_1\,d\tau, \qquad i,j=1,2;$$

$$z_i(s,t) = \psi_i(s) + \int_0^t \{k_1[p_{i11}(s,\tau)z_1(s,\tau) + p_{i12}(s,\tau)z_2(s,\tau)]$$
$$-\,k_2[p_{i12}(s,\tau)z_1(s,\tau) + p_{2i2}(s,\tau)z_2(s,\tau)]\}\,d\tau, \qquad i=1,2,$$

where

$$x_i(s,t) = \psi_i^{\xi}(s) + \int_0^t [k_1 v_{1i}(s,\tau) - k_2 v_{2i}(s,\tau)]\,d\tau, \qquad i=1,2,$$

$$p_{i11}(s,t) = -\int_0^{l_1} \varphi(\xi_1,\xi_2)\,\Pi_{i\xi_1}\Big|_{\substack{\tau=0\\ \xi_1=\psi_1(s)}} d\xi_2 + \iint_{\tau=0} \varphi_{\xi_1}(\xi_1,\xi_2)\,\Pi_{i\xi_1}\,d\xi_1 d\xi_2$$

$$-\,a_i^2 \int_0^t \int_0^{l_2} F_{i\xi_2}(\xi_2,\tau)\,\Pi_{i\xi_2}\Big|_{\xi_1=l_{i1}} d\xi_2 d\tau - a_i^2 \int_0^t \int_0^{l_2} F_{i\tau}(\xi_2,\tau)\,\Pi_i\Big|_{\xi_1=l_{i1}} d\xi_2 d\tau$$

$$-\,a_i^2 \int_0^{l_2} F_i(\xi_2,\tau)\,\Pi_i\Big|_{\substack{\tau=0\\ \xi_1=l_{i1}}} d\xi_2 - a_i^2 \int_0^t F_{i1}(\xi_1,\tau)\,\Pi_{i\xi_1}\Big|_{\xi_1=x_1(s,\tau)} d\tau$$

$$+a_i^2 \int_0^t \int_0^{l_1} F_{i1\xi_1}(\xi_1,\tau)\,\Pi_{i\xi_1}\Big|_{\xi_2=l_{12}} d\xi_1 d\tau + (-1)^i a_i^2 \int_0^t F_{i2}(\xi_1,\tau)\,\Pi_{i\xi_1}\Big|_{\substack{\xi_1=x_1(s,\tau)\\ \xi_2=l_{22}}} d\tau$$

$$+a_i^2 \int_0^t \int_0^{l_1} F_{i2\xi_1}(\xi_1,\tau)\,\Pi_{i\xi_1}\Big|_{\xi_2=l_{22}} d\xi_1 d\tau + a_i^2 \iint_{\xi_1=x_1(s,\tau),\,\xi_2=x_2(s,\tau)} v_{i1}(s,\tau)\,\Pi_{ix_1x_1}\,d\xi_2 d\tau$$

$$+\,a_i^2 \iint_{\xi_1=x_1(s,\tau),\,\xi_2=x_2(s,\tau)} v_{i2}(s,\tau)\,\Pi_{ix_1x_1}\,d\xi_1 d\tau, \qquad i=1,2,$$

and $p_{i12}(s,t)$ and $p_{i22}(s,t)$ are expressed in terms of $v_{ij}(s,t)$ analogously.

Theorem 2. For $0 \le t \le \widetilde{T} \le T$, where $\widetilde{T} > 0$ is estimated in terms of the initial data of the problem, the solution of system B exists and is unique.

From Theorems 1 and 2 there follows the existence and uniqueness of a regular classical solution of problem A on the interval $0 \le t \le \widetilde{T}$.

Theorem 3. The regular classical solution of problem A for $0 \le t \le \widetilde{T}$ is stable in the norm C with respect to perturbations of the initial data in C.

As in the one-dimensional case ([4]), the regular classical solution can be continued, with its properties preserved, to an interval $0 \le t \le \widehat{T}$, $\widehat{T} \le \widetilde{T} \le T$, on which the surface S does not intersect the faces $[x_1 = 0,\ 0 \le x_2 \le l_2,\ 0 \le t \le T]$, $[x_1 = l_1,\ 0 \le x_2 \le l_2,\ 0 \le t \le T]$.

Remark. The results set forth extend to the case of the multifront Stefan problem, as well as to the case of a nonlinear heat equation with linear principal part and nonlinear boundary conditions and conditions on the fronts.

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Note: Figure translations are in progress. See original paper for figures.