

ON THE EMBEDDING OF STONE-ČECH COMPACTIFICATIONS OF DISCRETE SPACES IN BICOMPACTS

1969

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.86615>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.83

MATHEMATICS

B. EFIMOV

ON THE EMBEDDING OF STONE-ČECH COMPACTIFICATIONS OF DISCRETE SPACES IN BICOMPACTS

(Presented by Academician P. S. Aleksandrov, April 9, 1969)

It is not difficult to show that ordered spaces, Fréchet-Urysohn spaces (sequential closure coincides with topological closure), and hereditarily normal spaces do not contain βN , however large their cardinality may be. On the other hand, the author has shown ^(5,6) that the continuum hypothesis is equivalent to the assertion that every nonmetrizable dyadic bicomcompact contains βN . Here a result is announced asserting, without special hypotheses of set theory, that every infinite bicomcompact having “not too many” disjoint open sets and “sufficiently large” weight contains, as a subspace, βN . By T_τ we denote the discrete space of cardinality $\tau \geq \aleph_0$. If $\tau = \aleph_0$, then $T_\tau = N$. By τ^+ , $\exp \tau$, and $\log \tau$ are denoted, respectively, the cardinal next after τ , the exponent of τ , and the logarithm of τ . (The least \mathfrak{t} such that $\exp \mathfrak{t} \geq \tau$.) By cX is denoted the cellularity of X , the supremum of the cardinalities of families of disjoint open subsets of X .

Main theorem. *Every bicomcompact X for which $cX \leq \tau$ and $wX > \exp \exp \exp \tau$ contains all extremally disconnected spaces of weight $\leq (\exp \tau)^+$, in particular βT_τ .*

First of all let us note that:

(α) *In order that a bicomcompact X contain βT_τ , it is necessary and sufficient that there exist a continuous mapping of X onto the Tikhonov cube $I^{\exp \tau}$.*

Thus, by (α), the problem posed reduces to the question of when a bicomcompact X is mapped onto the Tikhonov cube I^τ . A system of pairs of sets

$$\mathfrak{H}_\tau = \{S_\alpha = (A_\alpha^0, A_\alpha^1)\}, \quad \alpha \in B, \quad |B| = \tau$$

will be called a **dyadic system of cardinality** τ of the space X , if: a) A_α^0 and A_α^1 are zero-sets of X for all $\alpha \in B$; b) $A_\alpha^0 \cap A_\alpha^1 = \emptyset$ for all $\alpha \in B$; c) for any finite set $\alpha_1, \dots, \alpha_s \in B$ and any set i_1, \dots, i_s of zeros and ones,

$$A_{\alpha_1, \dots, \alpha_s}^{i_1, \dots, i_s} = A_{\alpha_1}^{i_1} \cap \dots \cap A_{\alpha_s}^{i_s} \neq \emptyset.$$

The concept of a dyadic system is a slight modification of an analogous concept proposed by P. S. Aleksandrov and V. I. Ponomarev ⁽¹⁾.

The following proposition holds:

(β) *In order that a bicomact X be continuously mapped onto I^τ , it is necessary and sufficient that there exist in X a dyadic system \mathfrak{H}_τ of cardinality τ .*

Thus, by (α) and (β), the problem posed reduces to the question of when a dyadic system \mathfrak{H}_τ is contained in a bicomact X . The answer to this question is closely connected with the nature of the number τ , and, if the construction of a dyadic system of maximal cardinality is in question, also with the hypotheses of set theory. Below we introduce two new cardinal in-

variants of topological spaces, “massiveness” and “tightness,” which make it possible to give an almost final solution of the question posed.

A system $\delta = \{\bar{U}\}$, consisting of canonically closed subsets of X , will be called a δ -system of the point $x \in X$ if: 1) $(x) = \bigcap \bar{U}$, $\bar{U} \in \delta$; 2) the system $\delta_0 = \{\text{int } \bar{U}\}$ is centered; and 3) for every open $V \ni x$ there exists $\bar{U} \in \delta$ such that $V \supset \bar{U}$. The minimum of the cardinalities of the δ -systems of the point x will be called the δ -character of $x \in X$ [$\delta(x, X)$]. The massiveness of a topological space X is defined by

$$mX = \min_{x \in X} \delta(x, X).$$

The tightness of X is defined by $eX = \sup(mY)$, where $Y \subset X$. For example, if $X = \beta T_\tau$, then $mX = 1$, while $eX = \exp \tau$. Note that always $mX \leq eX$.

Finally, we shall call a cardinal number τ \mathfrak{m} -unattainable if for every $\mathfrak{n} < \tau$ we have $\mathfrak{n}^{\mathfrak{m}} < \tau$. For example, $(\exp \mathfrak{m})^+$ is \mathfrak{m} -unattainable for every $\mathfrak{m} \geq \aleph_0$. Every dominant number τ is \mathfrak{m} -unattainable for every $\mathfrak{m} < \tau$. On the other hand, for example, \aleph_1 is not \aleph_0 -unattainable.

Theorem 1. *Every bicomact X of massiveness $\geq \tau \geq \aleph_0$ maps onto $I^{\log \tau}$.*

Theorem 2. *Let X be a bicomact, with $cX \leq \mathfrak{n}$, $mX \geq \tau$, and let τ be an \mathfrak{n} -unattainable cardinal. Then X maps continuously onto I^τ .*

Theorems 1 and 2 are proved analogously. For example, in the proof of Theorem 1, by transfinite induction we find in X a dyadic system $\mathfrak{S}_{\log \tau}$ of cardinality $\log \tau$. Here, if pairs $\{S_\alpha\}$, $\alpha < \beta$, have already been constructed, then in the process of selecting a pair S_β , “orthogonal” to the pairs S_α , $\alpha < \beta$, the following lemma plays an essential role.

Lemma 1. *Let $f : X \rightarrow I^{\mathfrak{m}}$ be a regular mapping of a bicomact X of massiveness $\geq \tau$ into $I^{\mathfrak{m}}$, with $\aleph_0 \leq \exp \mathfrak{m} < \tau$. Then there exists a disjoint pair (W_0, W_1) of zero-sets in X such that $W_i \cap f^{-1}y \neq \emptyset$ for all $y \in D^{\mathfrak{m}}$, $i = 1, 2$, where $D^{\mathfrak{m}}$ is the set of pseudovertrices of $I^{\mathfrak{m}}$.*

A mapping $f : X \rightarrow I^m$ is called regular if $f(X) \supset D^m$, where

$$D^m = \prod_{\alpha \in m} (O_\alpha, 1_\alpha),$$

with O_α and 1_α the end-points of the interval I_α . The set D^m is the set of “pseudovertrices” of I^m .* With the help of Theorems 1 and 2 one proves

Theorem 3. *If a bicompactum X maps onto I^τ , then $eX \geq \tau$. Conversely, if $eX \geq \tau$, then X maps onto $I^{\log(\tau^+)}$.*

The proof of the main theorem is based on Theorem 2 and on the following Lemma 2 “of Ramsey type,” which establishes a connection between massiveness, cellularity, and the Hausdorff cardinality of the space X .

Lemma 2. *Let X be a Hausdorff space, with $\delta(x, X) \leq n \leq \aleph_0$ for every point $x \in X$. If $|X| > \exp n$, then $cX \geq n^+$.*

The author has found a topological proof of this lemma, formally independent of combinatorial set theory. It should be noted, however, that the most “economical” proof of this lemma is obtained by combining the corresponding results of Erdős, Rado, and Hajnal (7) and of Juhász and Hajnal (8).

Corollary of the main theorem. *Let X be a bicompactum satisfying Suslin’s condition, and suppose that one of the following conditions is fulfilled: a) X is a Fréchet-Urysohn space; b) X is hereditarily normal; c) X is a χ -space in the sense of A. V. Arhangel’skii and V. I. Ponomarev (2).*

Then $wX \leq \exp \exp \exp \aleph_0$.

Let us note that examples of ordered bicompacta show that the restriction on cX in the main theorem is essential. No examples,

* There are no vertices in I^m , since by Keller’s theorem I^m is topologically homogeneous.

showing that the estimate wX is sharp are not known to the author, although for some particular classes of bicompacta it has been possible to lower this estimate. Let us further note that, if X belongs to one of the classes a), b), or c), then it follows from Theorem 3 that in X there exists a dense $M \subset X$ such that $\delta(x, X) \leq \exp \aleph_0 = \mathfrak{c}$ for all $x \in M$. Otherwise, $eX \geq \mathfrak{c}^+$, and, since $\log \mathfrak{c}^+ \geq \aleph_1$, there would exist a mapping of X onto I^{\aleph_1} , which is impossible. In connection with this the following question arises: does every bicompactum X belonging to one of the classes a), b), or c) satisfy the first axiom of countability at the points of some dense subset $M \subset X$?

If X is a zero-dimensional bicompactum, then in the statements of Theorems 1, 2, and 3, (α) and (β), one may replace I^τ by D^τ . For example:

Theorem 1-bis. *Every zero-dimensional bicompactum X of cardinality $\geq \tau \geq \aleph_0$ maps onto $D^{\log \tau}$.*

This theorem, proved without the generalized continuum hypothesis, is a more successful form of Theorem 1 from ⁽⁴⁾.

Theorem 4. *Let X be an extremally disconnected bicompactum and let \mathfrak{t} be an infinite cardinal number such that $\mathfrak{t} < \log \log(wX)$. Then X maps continuously onto $D^{\log(\mathfrak{t}^+)}$.*

Let us note that all the theorems proved can be formulated in the dual category of Boolean algebras. For example, the main theorem, taking into account (α) and (β) , can be formulated as follows. Let A be a Boolean algebra, with $cA \leq \tau$ and $|A| > \exp \exp \exp \tau$. Then A contains a free subalgebra having $(\exp \tau)^+$ generators. The cellularity cA is defined as the supremum of the cardinalities of families of disjoint elements of A . Let us note that all the indicated theorems are valid for locally bicompact spaces. Apparently, this is the maximal degree of generality in the present situation, since for every $\tau \geq \aleph_0$ there exist $(\Sigma$ -products ⁽³⁾) pseudocompact spaces satisfying Suslin's condition, of weight τ , and not mapping onto I^τ , and hence not containing βN . The author does not know the answer to the following question: does every infinite bicompactum contain either βN or the Aleksandrov compactification αN of the natural series?

Central Economic-Mathematical Institute
Academy of Sciences of the USSR
Moscow

Received
2 IV 1969

CITED LITERATURE

- ¹ P. S. Aleksandrov, V. I. Ponomarev, *Fund. Mat.*, **50**, 419 (1962).
- ² A. V. Arkhangel'skii, V. I. Ponomarev, *DAN*, **182**, No. 5, 993 (1968).
- ³ B. Efimov, *DAN*, **152**, No. 4 (1963).
- ⁴ B. Efimov, *DAN*, **178**, No. 3 (1968).
- ⁵ B. Efimov, *DAN*, **185**, No. 5 (1969).
- ⁶ B. Efimov, *DAN*, **187**, No. 1 (1969).
- ⁷ P. Erdos, A. Hajnal, R. Rado, *Acta Math. Acad. Sci. Hung.*, **16**, 93 (1965).
- ⁸ A. Hajnal, I. Juhasz, *Proc. Koninkl. Nederl. Akad. Wet.*, **A 70**, No. 3, 343 (1967).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.