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Abstract

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ON THE BOUNDARY VALUES OF GENERALIZED SOLUTIONS OF ELLIPTIC EQUATIONS

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It is known that if a function analytic in a disk has power growth when approaching the boundary, then generalized boundary values exist for it, from which the function can be recovered by means of the Poisson kernels. In the present paper it is shown that analogous properties are also possessed by generalized solutions of elliptic equations. The results obtained also strengthen the Lions–Magenes theorems on traces ([1–3]). The paper consists of two sections; the first is auxiliary in character, while the main assertions of the paper are contained in § 2.

§ 1. Let G be a bounded domain of n-dimensional space, and let Γ be its boundary. In $\overline{G} = G \cup \Gamma$ let a properly elliptic differential expression $L = L(x, D)$ of order $2m$, with complex coefficients, be given. For simplicity we shall assume that the coefficients of the expression L and the surface Γ are infinitely smooth. For an arbitrary real s we consider the spaces $W_p^s(G)$, $W_p^{s-1/p}(\Gamma)$; $\|u\|_{s,p}$, $\langle\!\langle u \rangle\!\rangle_{s-1/p,p}$ are the norms in these spaces (if $s \geq 0$ is an integer, then $W_p^s(G)$ is the space of S. L. Sobolev, while the spaces $W_p^s(G)$ and $W_{p'}^{-s}(G)$ $(1/p+1/p' = 1)$ are dual with respect to $(\cdot, \cdot) = (\cdot, \cdot)_{L_2(G)}$; the spaces $W_p^{s-1/p}(\Gamma)$ and $W_{p'}^{-(s-1/p)}(\Gamma)$ are also dual with respect to $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L_2(\Gamma)}$ (see ([4,5]); if s is not an integer, then these spaces are defined by complex interpolation ([4,6–9]).

Let l be an arbitrary integer. As in ([5,10–13]), denote by $\widetilde{W}_{p,2m}^{\,l}(G) = \widetilde{W}_p^{\,l}(G)$ the completion of $C^\infty(\overline{G})$ with respect to the norm

$$\|u\|_{\widetilde{W}_p^{\,l}(G)} = \left(\|u\|_{l,p}^p + \sum_{j=1}^{m} \langle\!\langle D_\nu^{j-1} u \rangle\!\rangle_{l-j+1-1/p,p}^p \right)^{1/p} \qquad (1)$$

$$\left(D_\nu = \frac{1}{i}\frac{\partial}{\partial \nu}, \quad \nu \text{ is the unit vector of the interior normal to } \Gamma \text{ at the point } x \right).$$

The closure S of the mapping

$$u \to (u|_G,\ u|_\Gamma, \dots, D_\nu^{2m-1} u|_\Gamma) \quad (u \in C^\infty(\overline{G}))$$

establishes an isometric correspondence between $\widetilde{W}_p^s(G)$ and a subspace of the direct sum

$$K_{l,p}^{(2m+1)} = W_p^l(G) \dot{+} \sum_{j=1}^{2m} W_p^{l-j+1-1/p}(\Gamma).$$

Here

$$S\widetilde{W}_p^l(G) = \{(u_0, u_1, \dots, u_{2m}) : u_0 \in W_p^l(G),\ u_j \in W_p^{l-j+1-1/p}(\Gamma)\ (j = 1, \dots, 2m);$$

for $l - j + 1 - \frac{1}{p} > 0$,

$$u_j = D_\nu^{j-1} u_0|_\Gamma\}.$$

For nonintegral s we define $\widetilde{W}_p^s(G)$ by means of complex interpolation between $\widetilde{W}_p^{[s]}(G)$ and $\widetilde{W}_p^{[s]+1}(G)$.

If $N(x, D)$ $(x \in \overline{G})$ is an arbitrary differential expression of order $r \leq 2m$ with sufficiently smooth coefficients, and $B(x, D)$ $(x \in \Gamma)$ is an arbitrary boundary differential expression of order $t \leq 2m - 1$ with sufficiently smooth coefficients, then

$$Nu\|_{s-r,p} \leq C_s \|u\|_{\widetilde{W}_p^s(G)}, \qquad \langle\!\langle Bu \rangle\!\rangle_{s-t-1/p,p} \leq C_s \|u\|_{\widetilde{W}_p^s(G)} \quad (u \in C^\infty(\overline{G})). \tag{2}$$

Therefore the closures N, B of the mappings $u \to Nu,\ u \to Bu|_\Gamma$ $(u \in C^\infty(\overline{G}))$ act continuously from all of $\widetilde{W}_p^s(G)$ respectively into $W_p^{s-r}(G)$, $W_p^{s-t-1/p}(\Gamma)$. In this (strong) sense, for arbitrary $u \in \widetilde{W}_p^s(G)$ the quantities $Nu \in W_p^{s-r}(G)$, $Bu|_\Gamma \in W_p^{s-t-1/p}(\Gamma)$ are defined ([5], [10–13]). The application of differential expressions to elements of $\widetilde{W}_p^s(G)$ may also be understood in another (weak) sense. By integration by parts we find

$$(Nu, v) = (u, N^+ v) + \sum_{j=1}^{r} \langle D_\nu^{j-1} u, R_{jv} \rangle \quad (u, v \in C^\infty(\overline{G})),$$

$$\langle Bu, v \rangle = \sum_{j=1}^{t+1} \langle D_\nu^{j-1} u, B^{(j)} v \rangle \quad (u, v \in C^\infty(\Gamma)); \tag{3}$$

here N^+ is the expression formally adjoint to N; $B^{(j)}$ are differential expressions on Γ, containing only tangential differentiations. If now $u \in \widetilde{W}_p^s(G)$, $Su = (u_0, u_1, \dots, u_{2m})$, then $Nu = f \in W_p^{s-r}(G)$ if and only if

$$(u_0, N^+v) + \sum_{j=1}^{r} \langle u_j, R_{jv} \rangle = (f, v) \quad (v \in C^\infty(\overline{G})); \tag{4}$$

similarly, $Bu|_\Gamma = \varphi \in W_p^{s-t-1/p}(\Gamma)$ if and only if

$$\sum_{j=1}^{t+1} \langle u_j, B^{(j)}v \rangle = \langle \varphi, v \rangle \quad (v \in C^\infty(\Gamma)). \tag{5}$$

Below $u|_G$ is the first component of the vector Su $(u \in \widetilde{W}_p^s(G))$.

§ 2. We now present the main assertions of this paper.

Theorem 1. *For every real s, the norms $\|u\|_{\widetilde{W}_p^s(G)}$ and*

$$\|u\|_{W_L^{s,p}(G)} = \|u\|_{s,p} + \|Lu\|_{s-2m,p} \tag{6}$$

are equivalent. Therefore $\widetilde{W}_p^s(G)$ coincides with the completion $W_L^{s,p}(G)$ of the set $C^\infty(\overline{G})$ in the norm (6). Moreover, $W_L^{s,p}(G)$ coincides with the set of all pairs (u_0, f), where $u_0 \in W_p^s(G)$, $f \in W_p^{s-2m}(G)$, and, in the sense of distribution theory, $Lu_0 = f$, i.e.

$$(u_0, L^+v) = (f, v) \quad \left(v \in \mathring{W}_{p'}^{2m-s}(G) \cap \mathring{W}_{p'}^{2m}(G), \quad 1/p + 1/p' = 1\right), \tag{7}$$

where L^+ is the expression formally adjoint to L, and $\mathring{W}_p^{2m}(G)$ is the closure in $W_p^{2m}(G)$ of the set of sufficiently smooth functions finite in G.

Thus every element $(u_0, f) \in W_L^{s,p}(G)$ can be identified in a natural way with the corresponding element $u \in \widetilde{W}_p^s(G)$; therefore (see § 1) for $u = (u_0, f)$ there exist $Nu \in W_p^{s-r}(G)$, $Bu|_\Gamma \in W_p^{s-t-1/p}(\Gamma)$. In particular, there exist $D_\nu^{j-1}u|_\Gamma = u_j \in W_p^{s-j+1-1/p}(\Gamma)$ $(j = 1, \dots, 2m)$, where $(u_0, u_1, \dots, u_{2m}) = Su$; the element u is a solution of the problem

$$Lu = f, \qquad D_\nu^{j-1}u|_\Gamma = u_j \quad (j = 1, \dots, m). \tag{8}$$

We also note that from inequalities (2) and Theorem 1 there follow the theorems on Lions-Magenes traces ([1]-[3]).

Transferring, for $u, v \in C^\infty(\overline{G})$, all differentiations from u to v by integration by parts, we easily find, by means of passage to the limit, that for every real s

$$(Lu, v)_G = (u|_G, L^+v) + \sum_{j=1}^{2m} \langle D_\nu^{j-1} u, T_{2m-j+1} v\rangle \quad \left(u \in \widetilde{W}_p^s(G),\ v \in \widetilde{W}_{p'}^{2m-s}(G)\right). \tag{9}$$

Let $*$ in Theorem 1 mean $s \geqslant 1$, $s > n/p$, and let $R_x = R(x, \cdot) = R(x, y)$ be the Green function of problem (8) (see ([11])); then the first component $u_0(x) = u|_G$ of the solution $u \in \widetilde{W}_p^s(G)$ of problem (8) can be found from the formula

$$u_0(x) = (f, \overline{R}_x) - \sum_{j=1}^{m} \langle u_j, T_{2m-j+1} \overline{R}_x\rangle + \tilde{u}(x); \tag{10}$$

here $x \in G \cup \Gamma$, $\tilde{u}(x) \in \mathfrak{N} = \{\omega \in C^\infty(\overline{G}) : L\omega = 0,\ D^{i-1}\omega|_\Gamma = 0\ (j = 1, \dots, m)\}$, $(u_0 - \tilde{u}, \mathfrak{N}) = 0$. It turns out that if inside G the function f is sufficiently smooth and x is an interior point of the domain, then formula (10) is valid for every real s.

Theorem 2. Let $u_0 \in W_p^s(G)$, $f \in W_p^{s-2m}(G)$, and, in the sense of distribution theory, $Lu_0 = f$. If $f \in W_p^{s+k-2m}(G_1)$ $(G_1 \subset G)$, where $k \geqslant 0$ and $s + k \geqslant 1$, $s + k > n/p$, then for every subdomain $G_0 \subset G_1$ such that $\overline{G}_0 \subset G_1$, $u_0 \in W_p^{s+k}(G_0)$, and formula (10) is valid for $x \in G_1$.

Suppose, for example, that $u_0(x) \in C^\infty(G)$ and $Lu_0(x) = 0$. Suppose that as one approaches Γ, $u_0(x)$ has a power-type singularity. Then one can define in a natural way the regularization u_∂ of the function $u_0(x)$; moreover $u_0 \in W_p^s(G)$ for some $s \leqslant 0$, depending on the order of the singularity of the function $u_0(x)$ near Γ, and $Lu_\partial = 0$. Therefore, for $u_0(x)$, formula (10) with $f = 0$ is valid inside G. Indeed, there exists a neighborhood G_2 in $\overline{G}$ of the surface Γ, through each point x of which passes a unique normal to Γ. Let $x' \in \Gamma$ be the base of the normal passing through the point $x \in G_2$, and let $\delta(x)$ be the distance between the points x and x'. If $u_0(x) = \omega(x)/\rho^\alpha(x)$, where $\omega(x)$ is a function bounded in $\overline{G}$, $\rho(x) \in C^\infty(G)$ is a positive function in G, equal to $\delta(x)$ in G_2, and $k \leqslant \alpha < k + 1$ (k is a natural number), then we define the regularization u_∂ of the function $u_0(x)$ as follows:

$$(u_0, v) = \int_{G \setminus G_2} u_0(x)\overline{v(x)}\, dx +$$

$$+ \int_{G_2} u_0(x) \left(\overline{v(x)} - \overline{v(x')} - \dots - \frac{1}{(k-1)!} \frac{\partial^{k-1}\overline{v(x')}}{\partial \nu^{k-1}} (\delta(x))^{k-1}\right) dx.$$

It is clear that

$$|(u_0, v)| \leqslant C\|v\|_{C^{k-1+\varepsilon}(G)} \leqslant C\|v\|_{t,p'},$$

where $\alpha - k < \varepsilon < 1$, and $t - n/p' > k - 1 + \varepsilon$; therefore $u_\partial \in W_p^{-t}(G)$ $(t > n - 1 + \alpha - n/p)$.

Let us now indicate the conclusion of the proof of the first assertion of Theorem 1. From inequality (2) it follows immediately that it is enough to establish the estimate

$$\|u\|_{\widetilde{W}_p^s(G)} \leqslant C\|u\|_{W_L^{s,p}(G)} \quad (u \in C^\infty(\overline{G})). \tag{11}$$

We note that, since the expressions $\{T_j(x, D)\}_{j=1}^{2m}$ form a Dirichlet system of order $2m$, it can be shown that for every vector

$$\psi = (\psi_1, \dots, \psi_{2m}) \in \sum_{j=1}^{2m} W_{p'}^{2m-s-j+1-1/p'}(\Gamma)$$

there exists an element $v \in \widetilde{W}_{p'}^{2m-s}(G)$ such that $T_{jv}|_\Gamma = \psi_j$ $(j = 1, \dots, 2m)$, and the operator $\psi \mapsto v$ is continuous from

$$\sum_{j=1}^{2m} W_{p'}^{2m-s-j+1-1/p'}(\Gamma)$$

to $\widetilde{W}_{p'}^{2m-s}(G)$.

Fix $u \in C^\infty(\overline{G})$ and consider the functional **

$$l(v) = (Lu, v) - (u, L^+ v) \quad (v \in \widetilde{W}_{p'}^{2m-s}(G)).$$

From (9) it follows easily that $l(v)$ depends only on the vector $\psi = (\psi_1, \dots, \psi_{2m})$, $\psi_j = T_{jv}|_\Gamma$; moreover

$$|l(v)| = |l_1(\psi)| \leqslant$$

* Below in this paragraph a problem formulated in a conversation with the author by S. D. Eidelman is studied.

** Cf. the proof in (1) of the trace theorems.

$$\ll C_1\|u\|_{W_{L,p}^s(G)}\|v\|_{\widetilde{W}_{p'}^{2m-s}(G)} \ll C_2\|u\|_{W_{L,p}^s(G)}\|\psi\|_{\sum_{j=1}^{m} W_{p'}^{2m-s-j+1-1/p'}(\Gamma)}.$$

Therefore there exist elements $\tau_j u \in W_p^{s-j+1-1/p}(\Gamma)$ $(j = 1, \dots, 2m)$ such that

$$(Lu, v) - (u, L^+ v) = \sum_{j=1}^{2m} \langle \tau_j u, T_{2m-j+1} v\rangle \quad (v \in \widetilde{W}_{p'}^{2m-s}(G)), \tag{12}$$

$$\left(\sum_{j=1}^{2m} \langle\!\langle \tau_j u \rangle\!\rangle_{s-j+1-1/p,p}^{p}\right)^{1/p} \ll C_2 \|u\|_{W_{L,p}^{s}(G)} \quad (u \in C^\infty(\overline{G})). \tag{13}$$

But from (9) and (12) it follows directly that $\tau_j u = D_\nu^{j-1} u|_\Gamma$; therefore, if s is an integer, then estimate (11) follows from (13) and (1). If s is not an integer, then from the homeomorphism theorem[5] it follows that

$$\|u\|_{\widetilde{W}_p^s(G)} \leq C \left(\|Lu\|_{s-2m,p} + \sum_{j=1}^{m} \langle\!\langle D_\nu^{j-1} u \rangle\!\rangle_{s-j+1-1/p,p} + \|u\|_{s,p} \right) \quad (u \in C^\infty(\overline{G})), \tag{14}$$

and from (14) and (13) estimate (11) again follows.

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Note: Figure translations are in progress. See original paper for figures.