EMBEDDING THEOREMS AND BEST APPROXIMATIONS

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Abstract

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EMBEDDING THEOREMS AND BEST APPROXIMATIONS

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1. The present note is a direct continuation of the author' s investigations (see ([2–5])). In the cited papers the corresponding historical information was indicated, and the necessary definitions and concepts pertaining to the circle of questions under consideration were also given. Below we shall adhere to the notation and concepts from paper ([5]).

In the present article we formulate a number of results concerning embedding theorems and, in addition, give several assertions relating to estimates of best approximations of functions, which can be derived from the embedding theorems obtained by us. In particular, we give (see (8)) an estimate of the best approximation $E_n^{(\nu)}(f)$ in terms of the best approximations $E_k^{(p)}(f)$ $(1 \leq p < \nu < \infty)$, and this estimate is, in a certain sense, unimprovable. The embedding theorems formulated below were obtained by us with the aid of methods of the metric theory of functions (see ([5])).

2. In this section we give results relating to embedding theorems for functions of one variable.

Theorem 1. *Let $\varphi(t)$ be a nonnegative nondecreasing function on $[0, \infty)$. Then for every function $f(x) \in L(0,1)$ the inequality holds*

$$\int_0^1 \varphi(|f(x)|)\,dx \leq \sum_{n=1}^{\infty} 2^{-n}\varphi(9 \cdot 2^n \omega_1(2^{-n}, f)) + 2\|f\|_1, \qquad (1)$$

where $\|\psi\|_p$ is the norm of the function $\psi(x)$ in the space $L^p(0,1)$, and $\omega_p(\delta,\psi)$ is the modulus of continuity (in L^p) of the function $\psi \in L^p(0,1)$ (see ([5])).

This assertion is a strengthening of one of our results from paper ([4]) (Theorem 2).

On the basis of Theorem 1 it is easy to verify that the following is true.

Corollary 1. a) *If, for some $\varepsilon > 0$, the modulus of continuity*

$$\omega_1(\delta, f) \leq (1/9 - \varepsilon)\delta \ln(1/\delta) \qquad (0 < \delta \leq \delta_0 < 1/3),$$

then $f \in l^L$, *i.e.*

$$\int_0^1 \exp|f(x)|\,dx < \infty.$$

b) *There exists a function* $f_0(x) \in L(0,1)$ *such that*

$$\omega_1(\delta, f_0) \le \delta \ln(1/\delta) \qquad (0 < \delta < 1/3),$$

but $f_0 \notin l^L$.

In paper ([5]) (Theorem 3) it was shown that Theorem 1 for the case $\varphi(t) = t^\nu$ $(1 < \nu < \infty)$ is, in a certain sense, unimprovable. However, for rapidly increasing functions $\varphi(t)$ such a question has not been studied. In particular, Corollary 1 shows that an unimprovable condition has not been found for the embedding

$$H_1^{\omega(\tau)} \subset l^L$$

and that this question is connected with finding the corresponding constant C $(1/9 \le C \le 1)$, which appears as the multiplier before $\delta \ln(1/\delta)$.

Theorem 2. *Let* $\varphi(t)$ *be a nonnegative nondecreasing function on* $[0, \infty)$. *Then for every function* $f(x) \in L(0,1)$ *the inequalities are valid*

$$\int_0^1 |f(x)|\varphi(|f(x)|)\,dx \leqslant 2\varphi(1)\|f\|_1 + 8\|f\|_1^2 \sum_{n=1}^{\infty} \frac{\varphi(n+1) - \varphi(n)}{n} +$$

$$+68 \sum_{n=1}^{\infty} [\varphi(n+1) - \varphi(n)]\,\omega_1\left(\frac{\|f\|_1}{n}, f\right); \tag{2}$$

$$\int_0^1 |f(x)|\varphi(|f(x)|)\,dx \leqslant 2\varphi(\|f\|_1)\|f\|_1 + 8\|f\|_1 \sum_{n=1}^{\infty} \frac{\varphi((n+1)\|f\|_1) - \varphi(n\|f\|_1)}{n} +$$

$$+68 \sum_{n=1}^{\infty} [\varphi((n+1)\|f\|_1) - \varphi(n\|f\|_1)]\omega_1\left(\frac{1}{n}, f\right). \tag{3}$$

From Theorem 2 it follows directly, for example:

Corollary 2. If $f(x) \in L(0,1)$, then

$$\int_0^1 |f| \ln(1 + |f|)\,dx \leqslant 2\|f\|_1(8 + \ln(1 + \|f\|_1)) + 68 \sum_{n=1}^{\infty} \frac{1}{n}\,\omega_1\left(\frac{1}{n}, f\right).$$

An inequality of this type also holds for functions of the form $|f| \ln^\alpha(1 + |f|)(\ln\ln(2 + |f|))^\beta$ for $\alpha > 0$ and $\beta \in (-\infty, \infty)$ (when $\alpha = 0$, the number $\beta > 0$).

Theorem 3. Let the numbers $1 \leqslant p < \nu < \infty$, $0 \leqslant \beta < \infty$, and a modulus of continuity $\omega(\delta)$ be given. Then, in order that the embedding

$$H_p^\omega(\delta) \subset L^\nu \ln^\beta L, \tag{4}$$

hold, it is necessary and sufficient that

$$\sum_{n=1}^{\infty} n^{\nu/p-2} \omega^\nu \left(\frac{1}{n}\right) \ln^\beta(n+1) < \infty. \tag{5}$$

The sufficiency of condition (5) for the embedding (4) was established by us in paper (5) (Remark 6), where the necessity of condition (5) for $\beta = 0$ was also proved (see Theorem 3 in (5)).

We note that in the case $\beta = 0$ the inequality* is valid

$$\|f\|_\nu \leqslant C(\nu) \left\{ \|f\|_1 + \left[\sum_{n=1}^{\infty} n^{\nu/p-2} \omega_p^\nu \left(\frac{1}{n}, f\right) \right]^{1/\nu} \right\}, \tag{6}$$

where $1 \leqslant p < \nu < \infty$, and the constant $C(\nu) < 2^{13}\nu$ (see (5), Theorem 1).

3. The inequalities of the types (1), (2), (3), and (6) formulated above make it possible to obtain estimates for moduli of continuity and best approximations of functions in various metrics. For example, the following holds (see (5), Remark 6).

Theorem 4. If $1 \leqslant p < \nu < \infty$ and the function $f(x) \in L^p(0, 2\pi)$**, then

$$\omega_\nu \left(\frac{1}{n}, f\right) \leqslant C(\nu, p) \left\{ \sum_{k=n+1}^{\infty} k^{\nu/p-2} \omega_p^\nu \left(\frac{1}{k}, f\right) \right\}^{1/\nu} \quad (n \geqslant 1), \tag{7}$$

$$E_n^{(\nu)}(f) \leqslant C(\nu, p) \left\{ n^{1/p-1/\nu} E_n^{(p)}(f) + \left[\sum_{k=n+1}^{\infty} k^{\nu/p-2} \left(E_k^{(p)}(f)\right)^\nu \right]^{1/\nu} \right\} \quad (n \geqslant 1), \tag{8}$$

where $E_n^{(q)}(f)$ is the best approximation (in $L^q(0, 2\pi)$) to the function $f \in L(0, 2\pi)$ by trigonometric polynomials of order not exceeding n.

* By $C(\alpha, \beta, ...)$ we denote positive constants depending only on the parameters involved and, generally speaking, different in different formulas.

** By $L^p(0, 2\pi)$ is denoted the set of 2π-periodic measurable functions whose p-th power of the modulus is integrable on $[0, 2\pi]$.

Remark 1. Inequalities of type (7) hold not only for moduli of continuity of the first order, but also for moduli of continuity of any order. This fact, as well as Theorem 4, was first formulated by us in [5] (Remark 6).

Remark 2. Earlier A. A. Konyushkov (see [1], p. 56) and S. B. Stechkin established the estimate

$$E_n^{(\nu)}(f) \le C(\nu;p)\left\{n^{1/p-1/\nu}E_n^{(p)}(f)+\sum_{k=n+1}^{\infty}E_k^{(p)}(f)\,k^{1/p-1/\nu-1}\right\}\quad (1\le p<\nu;\ n\ge 1). \tag{9}$$

In connection with this we can say that inequality (8) expresses the true essence of the matter more accurately (the estimate of $E_n^{(\nu)}(f)$ in terms of $E_k^{(p)}(f)$ $(1 \le p < \nu < \infty)$) than does inequality (9). This is confirmed by the fact that for any $a_k \downarrow 0$ and $1 \le p < \nu < \infty$ the relation

$$\left\{\sum_{k=n}^{\infty}k^{\nu/p-2}a_k^{\nu}\right\}^{1/\nu} \le 8\left(n^{1/p-1/\nu}a_n+\sum_{k=n+1}^{\infty}k^{1/p-1/\nu-1}a_k\right)\quad (n\ge 1),$$

is valid, and therefore (9) follows from (8). Further, in a number of cases inequality (8) gives an order-sharp estimate for $E_n^{(\nu)}(f)$ than inequality (9). Indeed, if, for example, we take

$$E_k^{(p)}(f) = O\{k^{-1/p+1/\nu}\ln^{-2}(k+1)\},$$

then from (8) it follows that

$$E_n^{(\nu)}(f) = O\{\ln^{1/\nu-2}(n+1)\}\quad (1<\nu<\infty,\ n\to\infty), \tag{10}$$

whereas inequality (9) leads only to the relation

$$E_n^{(\nu)}(f) = O\{\ln^{-1}(n+1)\},$$

which is worse in order than (10). Moreover, if we take

$$E_k^{(p)}(f) = O\{k^{-1/p+1/\nu}\ln^{-1}(k+1)\},$$

then inequality (9) gives no estimate at all for $E_n^{(\nu)}(f)$, whereas (8) leads to the relation

$$E_n^{(\nu)}(f) = O\{\ln^{-1+1/\nu}(n+1)\}. \tag{11}$$

Remark 3. Under the assumptions made in Remark 2 on $E_k^{(p)}(f)$, the estimates (10) and (11) for $E_n^{(\nu)}(f)$ are, generally speaking, unimprovable. Moreover, estimate (8) cannot be strengthened if $E_k^{(p)}(f)$ decreases sufficiently regularly. To formulate the corresponding results we introduce a definition: if $q \in [1, \infty)$ and a sequence $N_k \downarrow 0$, then by $A_q(\{N_k\})$ we denote the set of all those functions $f \in L^q(0, 2\pi)$ for each of which

$$E_k^{(q)}(f) = O(N_k) \quad (k \to \infty).$$

Let us also put

$$D_n^{(p,\nu)}(\{N_k\}) = n^{1/p-1/\nu} N_n + \left\{ \sum_{k=n+1}^{\infty} k^{\nu/p-2} N_k^{\nu} \right\}^{1/\nu} \qquad (1 \le p < \nu < \infty,\ n \ge 1).$$

For example, the following assertion is true: if $M_k \downarrow 0$ is an arbitrary sequence, $1 \le p < \nu_i < \infty$, and

$$F_k = k^{-\alpha} \ln^{-\beta}(k+2) \quad (k > k_0),$$

where β is any number from $(-\infty, \infty)$ for $\alpha \in (1/p - 1/\nu,\ 1/p)$, and $\beta > 1/\nu$ for $\alpha = 1/p - 1/\nu$, then in order that the embedding

$$A_p(\{F_k\}) \subset A_\nu(\{M_k\})$$

hold, it is necessary and sufficient that

$$D_n^{(p,\nu)}(\{F_k\}) = O(M_n) \quad \text{as } n \to \infty.$$

Remark 4. In estimate (8) the free term

$$n^{1/p-1/\nu} E_n^{(p)}(f) \equiv d_n(f)$$

under regular decrease of $E_n^{(p)}(f)$ (in particular, when

$$E_n^{(p)}(f) \le C E_{2n}^{(p)}(f)$$

) can be included under the summation sign. More precisely, in inequality (8), under regular decrease of $E_n^{(p)}(f)$, the right-hand side may be replaced by

$$C(\nu,p)\left\{\sum_{k=n}^{\infty} k^{\nu/p-2}\,[E_k^{(p)}(f)]^{\nu}\right\}^{1/\nu}. \tag{12}$$

It is not excluded that in the general case the term $d_n(f)$ in estimate (8) can be replaced by $E_n^{(p)}(f)$. This is partly confirmed by the fact that the following is true.

Theorem 4′. a) Let $f \in L(0,2\pi)$. Then

$$E_n^{(2)}(f) \le \left\{\sum_{k=n}^{\infty}[E_k^{(1)}(f)]^2\right\}^{1/2} \le E_n^{(1)}(f) + \left\{\sum_{k=n+1}^{\infty}[E_k^{(1)}(f)]^2\right\}^{1/2} \quad (n \ge 1).$$

b) Let $M_k \downarrow 0$ and $F_k \downarrow 0$ be arbitrary sequences, where $\{F_k\}$ is convex and

$$\sum_{k=1}^{n} kF_k = O(nF_n).$$

Then, in order that the embedding $A_1(\{F_k\}) \subset A_2(\{M_k\})$ hold, it is necessary and sufficient that

$$\left(\sum_{k=n}^{\infty} F_k^2\right)^{1/2} = O(M_n) \quad (n \to \infty).$$

It should be noted, however, that in the general case (for example, when $\nu < 2p$) the right-hand side of estimate (8) cannot be replaced by expression (12).

Below we shall need one more definition. Namely, if $F(t)$ is a nonnegative nonincreasing function on $[0,\infty)$ and $f(x) \in L(0,2\pi)$, put

$$\widetilde{E}_n(f)_{F(L)} = \inf_{T_n}\int_0^{2\pi} F(|f(x) - T_n(x)|)\,dx,$$

where $T_n \equiv T_n(x)$ is an arbitrary trigonometric polynomial of degree not exceeding n. Using this definition and inequality (3), it is easy to verify that the following holds.

Theorem 5. Let $\varphi(t)$ be a nonnegative nondecreasing function on $[0,\infty)$. Then, if $f(x) \in L(0,2\pi)$, then

$$\widetilde{E}_n(f)_{L\varphi(L)} \le C\left\{\left[\varphi(E_n^{(1)}(f)) + \sum_{k=0}^{n} D_k\right] E_n^{(1)}(f) + \sum_{k=n+1}^{\infty} D_{kE}k^{(1)}(f)\right\} \quad (n \ge 0),$$

where

$$D_k = \sum_{m=k+1}^{\infty} \frac{\varphi((m+1)\|f\|_1) - \varphi(m\|f\|_1)}{m} \quad (k = 0, 1, ...).$$

From this theorem there follows, for example (see also (3)),

Corollary 3. If $f(x) \in L(0, 2\pi)$, then (for $n = 0, 1, ...$)

$$\widetilde{E}_n(f)_{L\ln(1+L)} \le C\left\{[\ln(1 + E_n^{(1)}(f)) + \ln(2+n)]E_n^{(1)}(f) + \sum_{k=n+1}^{\infty} \frac{E_k^{(1)}(f)}{k}\right\},$$

and if $f = T_n$, then

$$I(T_n) \equiv \int_0^{2\pi} |T_n| \ln(2 + |T_n|)\, dx \le C\|T_n\|_1 \ln\big[(n+1)(2 + \|T_n\|_1)\big].$$

At the same time,

$$\sup_{\|T_n\|_1=1} I(T_n) \ge C' \ln(2+n) \quad (C' = \text{const} > 0).$$

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Note: Figure translations are in progress. See original paper for figures.