NONLINEAR EQUATIONS WITH MONOTONE OPERATORS

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Abstract

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NONLINEAR EQUATIONS WITH MONOTONE OPERATORS

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In the present paper, existence and uniqueness theorems are established for solutions of nonlinear integral equations of Hammerstein type in the Lebesgue spaces $L_{p,n}(G)$.

In works known to the author, in the study of the Hammerstein equation $u + Ahu = 0$, where A is a linear integral operator and h is a Nemytskii operator, it was required that the Hammerstein operator Ah be completely continuous; and, when this requirement was abandoned, it was assumed that A is a self-adjoint operator, $p \geq 2$, $\operatorname{mes} G < \infty$. In the present paper it has been possible to dispense both with these requirements and with some others. In the first part, a general existence and uniqueness theorem is established for systems of Hammerstein equations, and in the last part an analogous theorem for an abstract equation of Hammerstein type.

1. Let G be a measurable set of positive (finite or infinite) measure in s-dimensional Euclidean space; let $g_i(u_1, u_2, \dots, u_n, x)$ be real functions, continuous jointly in $(u_1, u_2, \dots, u_n)$ for almost every $x \in G$ and measurable in G with respect to x for fixed $(u_1, u_2, \dots, u_n)$, $-\infty < u_i < +\infty$ $(i = 1, 2, \dots, n)$. Consider the system of nonlinear integral equations

$$u_i(x) + \sum_{j=1}^{n} \int_G K_{ij}(x, y)\, g_j(u_1(y), u_2(y), u_3(y), \dots, u_n(y), y)\, dy \qquad (1)$$

under the assumption that each of the kernels $K_{ij}(x, y)$ generates a bounded integral operator B_{ij}:

$$B_{ij} v = \int_G K_{ij}(x, y) v(y)\, dy$$

from the space $L^q(G)$ into $L^p(G)$, $p > 1$, $p^{-1} + q^{-1} = 1$.

Theorem 1. *If the matrix $K = (K_{ij}(x, y))$ generates a positive operator from $L_{q,n}$ into $L_{p,n}$, i.e.*

$$f(v) = \sum_{i,j=1}^{n} \int_G \int_G K_{ij}(x, y) v_i(x) v_j(y)\, dx\, dy \geq 0, \qquad v_i \in L^q,$$

and the functions g_i satisfy the conditions

$$|g_i(u_1, u_2, \dots, u_n, x)| \leq a_i(x) + b \sum_{k=1}^{n} |u_k|^{p-1}, \qquad a_i(x) \in L^q,\ b > 0, \tag{2}$$

$$\sum_{i=1}^{n} [g_i(u_1, u_2, \dots, u_n, x) - g(v_1, v_2, \dots, v_n, x)](u_i - v_i) \geq 0 \tag{3}$$

for almost every $x \in G$ and arbitrary u_i and v_i,

$$u_i g_i(u_1, u_2, \dots, u_n, x) \geq \alpha |u_i|^p - \sum_{k=1}^{n} |\beta_{ik}(x)|\, |u_k|^p - |c_i(x)|, \tag{4}$$

where $a > 0,\ 1 \leq \gamma < p,\ \beta_{ik}(x) \in L^t,\ t = p/(p - \gamma),\ c_i(x) \in L(G)$, then system (1) has a solution belonging to $L_{p,n}(G)$. This solution is unique if K generates a strictly positive operator or if inequality (3) is strict.*

The proof uses the properties of the Nemytskii operator h, generated by the functions g_i (^{1}), the weak lower semicontinuity of the functional $f(v)$ in the space $L_{q,n}$, and the Browder-Minty lemma (^{2}, ^{3}).

Remark 1. The assertion of Theorem 1 remains valid if inequality (4) is replaced by the requirement: there exists a positive number such that, as soon as $\|u\| > \lambda$,

$$(hu, u) \geq 0,$$

where h is the Nemytskii operator generated by the functions $g_i(u_1, u_2, \dots, u_n, x)$, and (v, u) is the value of the linear functional $v \in L_{q,n}$ on the vector $u \in L_{p,n}$.

2. Let the kernel $K(x, y)$ generate a linear bounded positive operator B:

$$Bv = \int_G K(x, y) v(y)\, dy$$

from the space L^q into L^p, $p > 1,\ p^{-1} + q^{-1} = 1$, and let $g(u, x)$ be a function continuous in u for almost every $x \in G$ and measurable in G with respect to x for every $u \in (-\infty, +\infty)$. From Theorem 1 it follows

Theorem 2. If the function $g(u, x)$, nondecreasing in u for almost every $x \in G$, satisfies the condition

$$|g(u,x)| \leq a(x) + b|u|^{p-1}, \qquad a(x) \in L^q,$$

and there exists $\lambda > 0$ such that

$$\int_G g(u(x), x)u(x)\,dx \geq 0 \quad \text{for } \|u\| > \lambda,$$

then the nonlinear integral equation

$$u(x) + \int_G K(x,y)g(u(y),y)\,dy = 0$$

has a solution belonging to L^p. This solution is unique if $g(u, x)$ strictly increases with respect to u, or when the kernel $K(x, y)$ generates a strictly positive integral operator.

3. We shall say that a real Banach space E has property (π) if the following conditions are fulfilled: 1) E is reflexive; 2) there exists a monotone hemicontinuous operator A from E^* into E such that $A0 = 0$, Au is continuous at zero of the space, and $(Au, u) \geq c\|u\|^\alpha$, where $\alpha > 1$.

Examples of spaces with property (π) are reflexive Banach spaces E in which the norm in E^* is Fréchet differentiable. In such spaces the operator A can be given by the formula

$$Ax = \|x\|^\alpha \operatorname{grad}\|x\|, \quad \alpha > 0,\ x \neq 0,\ x \in E^*; \qquad A0 = 0,$$

since

$$(Ax, x) = \|x\|^{\alpha+1}; \qquad (Ax - Ay, x - y) \geq (\|x\| - \|y\|)(\|x\|^\alpha - \|y\|^\alpha) \geq 0.$$

Theorem 3. Let the Banach space E have property (π), let F be a monotone hemicontinuous and bounded operator from E into E^* such that

$$(Fu, u) \geq 0, \qquad \text{if } \|u\| > \lambda > 0,$$

and let B be a linear bounded and positive operator from E^* into E.

* That is, equality almost everywhere in x takes place only when $u_i = v_i$ for all i.

Then the equation

$$u + BFu = 0$$

has a solution belonging to E. This solution is unique when B is a strictly positive operator or when F is a strictly monotone operator.

Let us note that from this theorem follows Theorem 4.7, established in paper (^{4}) for a Hilbert space.

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