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Abstract

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VIBRATIONS OF A ROTATING DISK UNDER PARTIAL BRAKING

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Vibrations of an elastic disk under complete braking were considered by us in [1]. In the case of partial braking, the radial and tangential elastic displacements are coupled with one another through the rotational inertial force; therefore, the system of differential equations for the elastic displacements does not split into two independent equations, as in the case [1]. We have succeeded in reducing the solution of our system to Bessel' s equation. The stresses are represented by convergent series of trigonometric and Bessel functions; moreover, for the shear stress we were able to obtain a simple, readily surveyable formula.

Let a circular, isotropic, homogeneous, elastic disk of radius R and density ρ rotate about the central axis perpendicular to its plane with angular velocity ω. At some instant of time $t = 0$, a torque is applied to an absolutely rigid rim mounted on the disk, such that the angular velocity becomes equal to $\omega_0 < \omega$. In this case the material of the disk will be in a generalized plane stress state, independent of the polar angle φ, which it is convenient to consider in a moving polar coordinate system rigidly connected with the rim, so that its pole coincides with the center of the disk. Then the differential equations of motion will have the form [2, 3]

$$\frac{1}{c_2^2}\left[\frac{\partial^2 u_r}{\partial t^2} - \omega_0^2 r - 2\omega_0 \frac{\partial u_\varphi}{\partial t}\right] = \frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(u_r r)\right\};$$

$$\frac{1}{c_2^2}\left[\frac{\partial^2 u_\varphi}{\partial t^2} + 2\omega_0 \frac{\partial u_r}{\partial t}\right] = \frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(u_\varphi r)\right\}, \qquad (1)$$

where $c_1^2 = (\lambda^* + 2\mu)/\rho$, $c_2^2 = \mu/\rho$, $\lambda^* = 2\lambda\mu/(\lambda + 2\mu)$, and λ and μ are the Lamé coefficients.

Thus, in contrast to complete braking ($\omega_0 = 0$), the radial and tangential displacements are coupled with one another. This creates additional difficulties in solving system (1). The indicated system must be solved under the following initial and boundary conditions:

$$u_r(r,0) = f(r) = \omega^2 r(R^2 - r^2)/8c_1^2$$

($f(r)$ is the value of u_r in a disk rotating uniformly with angular velocity ω):

$$\frac{\partial u_\varphi(r,0)}{\partial t} = \begin{cases} (\omega - \omega_0)r = \omega_1 r, & 0 \leqslant r < R, \\ 0, & r = R, \end{cases} \qquad u_\varphi\big|_{t=0} = \partial u_\varphi/\partial t\big|_{t=0} = 0 \quad (2)$$

$$u_r(R,t) = u_\varphi(R,t) = 0. \tag{3}$$

Applying to (1) the one-sided Laplace transform, and taking (2) into account, we obtain a system of equations for the L-images of the sought functions [4]:

$$\frac{1}{c_1^2}\left[-pf(r) + p^2\bar{u}_r - \frac{\omega_0^2 r}{p} - 2\omega_0 p\bar{u}_\varphi\right] = \frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}(\bar{u}_r r)\right\},$$

$$\frac{1}{c_2^2}\left[p^2\bar{u}_\varphi - \omega_1 r - 2\omega_0 f'(r) + 2\omega_0 p\bar{u}_r\right] = \frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}(\bar{u}_\varphi r)\right\}. \tag{4}$$

Expressing $\bar{u}_\varphi$ in terms of $\bar{u}_r$ from the first equation (4) and substituting it into the second, we obtain a differential equation with variable coefficients for $\bar{u}_r$,

$$D^2\bar{u}_r - \frac{p^2(c_1^2 + c_2^2)}{c_1^2 c_2^2}D\bar{u}_r + \frac{p^2(4\omega_0^2 + p^2)}{c_1^2 c_2^2}\bar{u}_r = \frac{p\omega^2 r}{c_1^4} + \frac{p(p^2 + 4\omega_0^2)\omega^2 r(R^2 - r^2)}{8c_1^4 c_2^2} + \frac{p(\omega_0^2 + 2\omega_1\omega_0)}{c_1^2 c_2^2}r, \tag{5}$$

where D is the differential operator

$$D\bar{u}_r = \frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}(\bar{u}_r r)\right\}.$$

Consider the homogeneous equation

$$D^2\bar{u}_r - aD\bar{u}_r + b\bar{u}_r = 0, \tag{6}$$

$$a = p^2(c_1^2 + c_2^2)/c_1^2 c_2^2, \qquad b = p^2(4\omega_0^2 + p^2)/c_1^2 c_2^2.$$

We write the equation in the form

$$D\bar{u}_r + \varkappa\bar{u}_r = 0, \tag{7}$$

where $\varkappa$ is a coefficient to be determined. Applying the operation D to equation (7), we obtain

$$D^2 u_r + \varkappa D \bar{u}_r = 0. \tag{8}$$

Eliminating Du_r from (7) and (8), we obtain:

$$D^2 \bar{u}_r - \varkappa^2 \bar{u}_r = 0. \tag{9}$$

Multiplying now (7) by a and subtracting from (9), we have

$$D^2 \bar{u}_r - aD\bar{u}_r - (\varkappa^2 + a\varkappa)\bar{u}_r = 0. \tag{10}$$

Comparing (10) with (6), we find $a\varkappa + \varkappa^2 = -b$, whence

$$\varkappa_{1,2} = -a/2 \pm \sqrt{a^2/4 - b}.$$

Thus it is sufficient for us to find the solution of the equation

$$D\bar{u}_r + \varkappa_i \bar{u}_r = 0 \qquad (i = 1, 2), \tag{11}$$

which reduces to Bessel's equation. A particular solution of the nonhomogeneous equation is found without difficulty.

Knowing $\bar{u}_r$ and using the first equation (4), we find $\bar{u}_\varphi$. The expressions for the originals follow from the inversion formula for the L-transform, using the residue theorem (^{4}).

$$u_r = \sum_{k=1}^{\infty} 2 \left\{ \frac{a_k^2(\omega - \omega_0)^2 + c_2^2(\omega_0^2 - \omega^2)\alpha_k^2}{a_k \alpha_k^2 \left[\alpha_k^2(c_1^2 + c_2^2) - 2a_k^2 + 4\omega_0^2\right]} \cos a_k t - \right.$$

$$\left. - \frac{(\omega_0^2 - \omega^2)\alpha_k^2 c_2^2 + (\omega - \omega_0)^2 b_k^2}{\alpha_k b_k^2 \left[2b_k^2 - \alpha_k^2(c_1^2 + c_2^2) - 4\omega_0^2\right]} \cos b_k t \right\} \frac{J_1(\alpha_k r)}{J_0(\alpha_k R)} + \frac{\omega_0^2 r (R^2 - r^2)}{8c_1^2},$$

$$u_\varphi = \frac{1}{2\omega_0} \frac{\partial u_r}{\partial t} + \frac{c_1^2}{\omega_0} \sum_{k=1}^{\infty} \left\{ \frac{a_k \left[a_k^2(\omega - \omega_0)^2 + c_2^2(\omega_0^2 - \omega^2)\alpha_k^2\right]}{a_k^3[\alpha_k^2(c_1^2 + c_2^2) - 2a_k^2 + 4\omega_0^2]} \sin a_k t - \right.$$

$$\left. - \frac{\alpha_k[(\omega_0^2 - \omega^2)\alpha_k^2 c_2^2 + (\omega - \omega_0)^2 b_k^2]}{b_k^3[2b_k^2 - \alpha_k^2(c_1^2 + c_2^2) - 4\omega_0^2]} \sin b_k t \right\} \frac{J_1(\alpha_k r)}{J_0(\alpha_k R)}. \tag{12}$$

where α_k are the roots of the equation

$$J_1(\alpha_k R) = 0,$$

$J_1(\alpha_k r)$ and $J_0(\alpha_k R)$ are Bessel functions of the first kind,

$$a_k = \sqrt{\frac{\alpha_k^2(c_1^2 + c_2^2) + 4\omega_0^2 - \sqrt{\alpha_k^4(c_1^2 - c_2^2)^2 + 8\omega_0^2\alpha_k^2(c_1^2 + c_2^2) + 16\omega_0^4}}{2}},$$

$$b_k = \sqrt{\frac{\alpha_k^2(c_1^2 + c_2^2) + 4\omega_0^2 + \sqrt{\alpha_k^4(c_1^2 - c_2^2)^2 + 8\omega_0^2\alpha_k^2(c_1^2 + c_2^2) + 16\omega_0^4}}{2}}$$

are the natural frequencies.

Let us note that the expression for u_r consists of two terms: the first describes the elastic vibrations of the disk as a result of braking, and the second describes the elastic displacements of points of the disk under uniform rotation with angular velocity ω. The presence of the displacement u_φ is entirely caused by partial braking. Under uniform rotation it is absent.

We now investigate the stressed state of the disk. The expressions for σ_{rr} and $\sigma_{\varphi\varphi}$ are represented by series converging as $1/\alpha_k^2$ and $1/\alpha_k^3$.

As for the series for $\tau_{r\varphi}$, it consists of two parts, one of which converges as $1/\alpha_k^2$, and the other as $1/\alpha_k$.

Using the asymptotic expressions for the Bessel functions and the formulas for trigonometric series (5), we write the expression for $\tau_{r\varphi}$

$$\tau_{r\varphi} = \frac{\mu(\omega - \omega_0)R\sqrt{R}}{\pi c_2\sqrt{r}} \left\{ \frac{\pi(c_2t + r)}{2R} \cos\frac{\pi}{4R}(c_2t + r - R) \right.$$

$$+ \sin\frac{\pi}{4R}(c_2t + r - R)\ \ln 2\cos\frac{\pi(c_2t + r)}{2R} + \frac{\pi}{2R}(c_2t - r)\sin\frac{\pi}{4R}(R + r - c_2t)$$

$$\left. + \cos\frac{\pi}{4R}(c_2t - r - R)\ \ln 2\cos\frac{\pi(c_2t - r)}{2R} + S(r,t) \right\}$$

$$\left(0 \le t < \frac{R - r}{c_2}\right);$$

$$S(r,t) \text{ is a series converging as } \sum_k O\left(\frac{1}{k^2}\right)$$

$$\tau_{r\varphi} = \frac{\mu(\omega - \omega_0)R\sqrt{R}}{\pi c_2 \sqrt{r}} \left\{ \frac{\pi[c_2 t + r - 2nR]}{2R} \cos \frac{\pi(c_2 t + r - R)}{4R} - \right.$$

$$- \sin \frac{\pi}{4R}(c_2 t + r - R) \cdot \frac{1}{2} \ln \frac{1}{2\left[1 + \cos \frac{\pi}{R}(c_2 t + r)\right]} +$$

$$+ \frac{\pi[c_2 t - r - 2nR]}{2R} \sin \frac{\pi(R + r - c_2 t)}{4R} -$$

$$\left. - \cos \frac{\pi}{4R}(c_2 t - r - R) \cdot \frac{1}{2} \ln \frac{1}{2\left[1 + \cos \frac{\pi}{R}(c_2 t - r)\right]} + S(r, t) \right\};$$

$$(2n - 1)/c_2 < t < [(2n + 1)R - r]/c_2 \quad (n = 1, 2, \ldots).$$

For

$$\frac{R - r}{c_2} < t < \frac{R + r}{c_2}$$

$$\frac{\pi[c_2 t - r - 2nR]}{2R}$$

must be replaced by

$$\frac{\pi}{2R}(c_2 t - r),$$

and

$$-\frac{1}{2} \ln \frac{1}{2\left[1 + \cos \frac{\pi}{R}(c_2 t - r)\right]}$$

by

$$\ln \left[2 \cos \frac{\pi(c_2 t - r)}{2R} \right].$$

At each point of the disk, at certain instants of time

$$t_k = \frac{(2k - 1)R + r}{c_2}$$

$(k = 1, 2, \ldots)$, the stress $\tau_{r\varphi}$ becomes infinite. Thus there occurs the so-called "phase impact." The time interval between two successive "phase impacts" at any point of the disk is

$$t_{k+1} - t_k = 2R/c_2.$$

Thus, in the case of partial braking an infinitely large tangential stress arises. This indicates that, under real conditions, tangential stresses may be of an order higher than σ_{rr} and $\sigma_{\varphi\varphi}$.

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References

1. P. Ya. Bershtein, B. I. Kogan, *Abstracts of Reports, Third All-Union Congress on Theoretical and Applied Mechanics*, Moscow, 1968.
2. I. N. Sneddon, D. S. Berry, *Classical Theory of Elasticity*, Moscow, 1961.
3. L. G. Loitsyanskii, A. I. Lur' e, *Course of Theoretical Mechanics*, Moscow-Leningrad, 1948.
4. H. Karslow, D. Eger, *Operational Methods in Applied Mathematics*, IL, 1948.
5. I. S. Gradshteyn, I. M. Ryzhik, *Tables of Integrals, Sums, Series, and Products*, Moscow, 1962.

Note: Figure translations are in progress. See original paper for figures.