EXISTENCE OF PERIODIC MOTIONS AND MOTIONS OF SPECIAL TYPE OF HAMILTONIAN SYSTEMS WITH A SMALL PARAMETER

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Abstract

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EXISTENCE OF PERIODIC MOTIONS AND MOTIONS OF SPECIAL TYPE OF HAMILTONIAN SYSTEMS WITH A SMALL PARAMETER

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A conservative dynamical system with n degrees of freedom is considered, defined by the Hamiltonian equations of motion

$$\frac{d}{dt}p = -\frac{\partial H}{\partial q}, \qquad \frac{d}{dt}q = \frac{\partial H}{\partial p} \qquad (p = p_1, \dots, p_n;\ q = q_1, \dots, q_n) \tag{1}$$

The Hamiltonian function $H = H(p, q, \mu)$ is assumed to be periodic in the variables $q_1, \dots, q_n$ with period 2π. System (1) is written in the variables p, q (action—angle). Such systems of differential equations occur in mechanics ([1–4]).

Recently a number of new important results have been obtained in proving the existence of almost-periodic ([5]) and periodic ([6]) motions. In the present paper another method is proposed, by which one can obtain results close to ([6]), formulated below in Theorems 1 and 4. Theorem 1 can be obtained simply from Theorem 2 of ([6]). Theorem 4 is not contained in ([6])*.

The phase space of system (1) is the toroidal space $E_n \times T_n$—the direct product of the n-dimensional Euclidean space E_n $(p \in E_n)$ by the n-dimensional torus T_n. Points on the torus T_n are described by Cartesian coordinates $q_1, \dots, q_n$, with two points $(q_1, \dots, q_n)$ and $(\tilde{q}_1, \dots, \tilde{q}_n)$ regarded as identical if the differences $q_i - \tilde{q}_i$ $(i = 1, \dots, n)$ are integer multiples of 2π.

Let us formulate the conditions imposed on system (1).

A. The Hamiltonian function has the form $H = H_0(p) + \mu H_1(p, q, \mu)$, where μ $(\mu \geqslant 0)$ is a small parameter.

Thus, system (1) is close to the integrable system

$$\frac{d}{dt}p = 0, \qquad \frac{d}{dt}q = \omega(p) \qquad \left(\omega(p) = \frac{\partial H_0}{\partial p}\right). \tag{2}$$

B. The Hessian of the function $H_0(p)$ at some fixed point $p_0 \in E_n$ is different from zero:

$$\det \left\| \frac{\partial^2 H_0}{\partial p^2} \right\|_{p=p_0} \neq 0.$$

C. The functions $H_0(p)$ and $H_1(p, q, \mu)$ are defined for values of p from some neighborhood of the point p_0, all $q \in E_n$, and nonnegative values of μ from some neighborhood of zero. These functions, with respect to the variables $p = (p_1, \dots, p_n)$, $q = (q_1, \dots, q_n)$, have all possible derivatives up to and including the 4th order, continuous in the aggregate of variables p, q, μ.

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* After the article had been submitted for publication, the author turned to V. I. Arnold for clarification; he kindly pointed out that Theorem 4 can also be obtained by the methods of paper ([6]). V. I. Arnold also reported that the questions of existence of periodic motions of a Hamiltonian system in a neighborhood of a given periodic solution of stable type are the subject of papers ([8–10]). The author expresses sincere gratitude to V. I. Arnold for valuable remarks.

The main result is contained in Theorems 1 and 4.

Theorem 1. *Suppose that conditions A, B, C are fulfilled, and let τ be an arbitrary positive number. Then there exists a sufficiently small positive number μ_0 ($\mu_0 = \mu_0(\tau)$) such that, for every $\mu < \mu_0$, system (1) on the interval $[0, \tau]$ has at least $n + 1$ distinct solutions p, q possessing the following properties:*

1. $|p - p_0| \leq \mu c$, where the constant c does not depend on $t \in [0, \tau]$ or on $\mu \in [0, \mu_0)$.
2. $p(\tau) = p(0)$, $q(\tau) = q(0) + \omega(p_0)\tau$.

If $\omega(p_0)\tau \equiv 0 \pmod{2\pi}$, then Theorem 1 implies the existence of at least one periodic trajectory of system (1).

The general plan of the proof of Theorem 1 is as follows. We investigate the extremal properties of a functional naturally generated by system (1). The existence of manifolds filled with extrema of this functional is established. It is then proved that some of the extrema are solutions of system (1).

The proof of Theorem 1 is connected with the consideration of the functional

$$I(p, q) = \frac{1}{\tau} \int_0^\tau \left[\left| \frac{d}{dt} p + \frac{\partial H}{\partial q} \right|^2 + \left| \frac{d}{dt} q - \frac{\partial H}{\partial p} \right|^2 \right] dt, \tag{3}$$

where the real vector-functions p, q satisfy the conditions*:

1. p, q are twice continuously differentiable vector-functions on $[0, \tau]$.

2. $|p - p_0| < c_0$, where c_0 is a certain sufficiently small positive constant, determined by the Hamiltonian H of system (1).
3. Each of the vector-functions p and $q - \omega(p_0)t$ assumes equal values at the endpoints of the interval $[0, \tau]$.
4. The mean value of the vector-function $q - \omega(p_0)t$ is equal to the prescribed vector
$$\frac{1}{\tau}\int_0^\tau (q - \omega(p_0)t)\, dt = q_0.$$

Theorem 2. *Suppose that the conditions of Theorem 1 are fulfilled, and that $\mu_0 = \mu_0(\tau)$ is a sufficiently small positive number. Then, for $\mu < \mu_0$, the lower bound of the functional (3) in the class of vector-functions satisfying conditions 1-4 is attained on a certain pair of vector-functions p, q. For these vector-functions the estimate*

$$|p - p_0| + |q - \omega(p_0)t - q_0| \leq \mu c_1$$

is valid, where the number c_1 does not depend on t or on $\mu \in [0, \mu_0)$.

A pair of vector-functions $p = p(t, \mu, q_0)$, $q = q(t, \mu, q_0)$ that delivers a minimum of the functional I in the class of functions satisfying conditions 1-4 will be called an extremal. The solutions of system (2), $p = p_0$, $q = \omega(p_0)t + q_0$, as the point q_0 runs over the torus T_n, fill an n-dimensional torus $T(0)$ in the phase space $E_n \times T_n$. These solutions coincide with the extremals of the functional (3) for $\mu = 0$.

All possible values of the extremal $p(t, \mu, q_0)$, $q(t, \mu, q_0)$, for fixed $t \in [0, \tau]$ and as the vector q_0 varies on the torus T_n, fill in the space $E_n \times T_n$ a manifold which we denote by $T(\mu, t)$.

Theorem 3. *Suppose that the conditions of Theorem 1 are fulfilled and $\mu < \mu_0$, where μ_0 is a sufficiently small positive number. The mapping Φ of the set $T(0)$ onto the set $T(\mu, t)$, established by the correspondence of the points*

$$(p(t, 0, q_0), q(t, 0, q_0)) \quad \text{and} \quad (p(t, \mu, q_0), q(t, \mu, q_0))$$

in the space $E_n \times T_n$, is a homeomorphism. The mapping Φ is continuous jointly in the variables q, μ. For fixed μ, the vector-function Φ with respect to the variables q satisfies a Lipschitz condition.

* If a and b are vectors in E_n, then $(a, b) = \sum_{i=1}^{n} a_i b_i$ and $|a| = \sqrt{(a, a)}$.

The proofs of Theorems 2 and 3 are not given here. Using Theorems 2 and 3, we outline the proof of Theorem 1.

Along the extremal $p(t, q_0), q(t, q_0)$, define the vector-functions $x(t, q_0) = \frac{d}{dt}p + \frac{\partial H}{\partial q}$ and $y(t, q_0) = \frac{d}{dt}q - \frac{\partial H}{\partial p}$. From the Euler equations for I one obtains the estimate

$$|x - x_{\text{cp}}| + |y| \leqslant \mu c_2 |x_{\text{cp}}|, \quad \text{where} \quad x_{\text{cp}}(q_0) = \frac{1}{\tau}\int_0^\tau x(t, q_0)\, dt$$

and the constant c_2 does not depend on t, μ, q_0. Thus, if $x_{\text{cp}}(q_0) = 0$, then the extremal p, q is a motion of system (1) satisfying the conditions of Theorem 1.

Let $q_0 = q_0(l)$ $(0 \leq l \leq 1)$ be a closed curve on the torus T_n. Let $p(t, q_0(l)), q(t, q_0(l))$ be an extremal of the functional (3). Extend the functions p, q beyond the interval $(0, \tau)$ so that the functions $p, q - \omega(p_0)t$ are periodic. Let $\hat{p}(t, q_0(l), s), \hat{q}(t, q_0(l), s)$ be a solution of system (1) which at the initial moment $t = 0$ coincides with the point $p(s, q_0(l)), q(s, q_0(l))$.

The quantities

$$I_1 = \int_0^\tau ds \int_0^1 (p(t+s, q_0(l)), \delta q(t+s, q_0(l))),$$

$$I_2 = \int_0^\tau ds \int_0^1 (\hat{p}(t, q_0(l), s), \delta\hat{q}(t, q_0(l), s)),$$

where δq and $\delta\hat{q}$ are the differentials of q and $\hat{q}$ with respect to l, do not depend on t. I_1 does not depend on t, since $\varphi = \int_0^1 (p, \delta q)$ is a periodic function of its argument. I_2 does not depend on t, since $\int_0^1 (\hat{p}, \delta\hat{q})$ is a Poincaré invariant, which does not depend on t.

Computing the quantity $\frac{d}{dt}(I_2 - I_1)\Big|_{t=0}$ leads to the equality

$$\int_0^\tau ds \int_0^1 [(x(s, q_0(l)), \delta q(s, q_0(l))) - (y(s, q_0(l)), \delta p(r, q_0(l)))] = 0.$$

Hence it follows that

$$\psi(q_0^{(2)}) = \int_{q_0^{(1)}}^{q_0^{(2)}} \left\{ \int_0^\tau [(x(t, q_0), \delta q(t, q_0)) - (y(t, q_0), \delta p(t, q_0))]\, dt \right\},$$

where the outer integral is taken along any curve on the torus T_n joining the point $q_0^{(1)}$ to the point $q_0^{(2)}$, defines a single-valued function $\psi(q_0^{(2)})$ on the torus T_n. The function on the torus has stationary points (7), at which $d\psi(q_0) = 0$:

$$\int_0^\tau [(x(t, q_0), \delta q(t, q_0)) - (y(t, q_0), \delta p(t, q_0))]\, dt = 0 \tag{4}$$

It can be shown that as $\mu \to 0$ the following representations hold for $x, y, \delta p, \delta q$, uniformly in $q_0 \in T_n$:

$$\begin{aligned} x &= x_{\text{cp}} + o(|x_{\text{cp}}|), \qquad y = o(|x_{\text{cp}}|), \\ \delta p &= o(|\delta q_0|), \qquad \delta q = \delta q_0 + o(|\delta q_0|). \end{aligned} \tag{5}$$

From (4) and (5) it follows that at the stationary points of the function $\psi(q_0)$ the vector $x_{\text{cp}}(q_0) = 0$. This completes the proof of Theorem 1.

Let us consider the case $\omega(p_0)\tau \equiv 0 \pmod{2\pi}$ and $|\omega(p_0)| \neq 0$. It is readily established that extrema of the functional I then, and only then, have common points in the space $E_n \times T_n$ when the mean values of the functions $q - \omega(p_0)t$ differ by a vector that is a multiple of $\omega(p_0)$. The function $\psi(q_0)$ has the property

$$\psi(q_0 + \omega(p_0)s) = \psi(q_0),$$

where s is any number. The function

$$\psi(q_0 - (q_0, \omega(p_0))\, \omega(p_0) |\omega(p_0)|^{-2})$$

may be regarded as a function defined on a torus of dimension $n-1$. Different stationary points of this function (there are at least n such points) will correspond to periodic solutions of system (1) which describe geometrically distinct trajectories in the phase space $E_n \times T_n$. Thus, the following theorem holds.

Theorem 4. *Suppose that conditions A, B, C are satisfied and*

$$\omega(p_0)\tau \equiv 0 \pmod{2\pi}, \qquad |\omega(p_0)| \neq 0,$$

where τ is a prescribed positive number. Then, for every sufficiently small μ, system (1) has at least n geometrically distinct τ-periodic trajectories. The solutions p, q that describe these trajectories satisfy the conditions $|p - p_0| \leq \mu c$, where the constant c does not depend on μ.

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Note: Figure translations are in progress. See original paper for figures.