EMBEDDING THEOREMS FOR FUNCTIONS DEFINED IN UNBOUNDED DOMAINS, AND THEIR APPLICATION TO THE SPECTRAL THEORY OF ELLIPTIC SELF-ADJOINT OPERATORS

MATHEMATICS

1969

SovietRxiv

Abstract

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UDC 517.514

MATHEMATICS

EMBEDDING THEOREMS FOR FUNCTIONS DEFINED IN UNBOUNDED DOMAINS, AND THEIR APPLICATION TO THE SPECTRAL THEORY OF ELLIPTIC SELF-ADJOINT OPERATORS

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(Presented by Academician I. M. Vinogradov on 26 VI 1968)

As K. Clark showed (1), the embedding $W_2^1(\Omega)$ into $L_2(\Omega)$ is not completely continuous if the unbounded domain Ω has infinite measure. A consequence of this is the non-discreteness of the spectrum of the Laplace operator in a domain containing a countable number of identical nonintersecting squares (see (2)).

In this paper, for a sufficiently general unbounded domain Ω, we consider the class $W_{p,\alpha}^l(\Omega, g)$, which coincides with the class $W_p^l(\Omega)$ in the case of a bounded domain (for $\Omega = E_n$, $W_{p,\alpha}^l(\Omega, g)$ is the class of L. D. Kudryavtsev $W_{p,\alpha}^l(E)$ (3–5)), and it is shown that the embedding $W_{p,\alpha}^l(\Omega, g)$ into $W_{p,\alpha_k}^k(\Omega, g)$ is completely continuous $(0 \leq k < l)$ for a certain exactly determined exponent of the weight degree α_k. With the help of this result an operator $a(X, D)$ is constructed, differing from the elliptic self-adjoint operator $L(X, D)$ by a positive factor of power growth at infinity, and it is proved that the operator $a(X, D)$ has a discrete spectrum in an unbounded domain not containing a sphere of positive radius.

Let the unbounded domain $\Omega \subset E_n$, $n \geq 1$, let g be a bounded interior subdomain of Ω, let α be a real number, and $1 \leq p \leq \infty$.

We introduce classes of functions as linear normed spaces of functions for which the following norms are meaningful and finite:

$$|f; L_{p,\alpha}(\Omega)| = |(1 + |X|)^{-\alpha} f;\ L_p(\Omega)|,$$

$$|f; L_{p,c(X)}(\Omega)| = |c^{1/p}(X) f;\ L_p(\Omega)|,$$

$$|f; W^l_{p,\alpha}(\Omega, g)| = |f; L_p(g)| + \sum_{|r|=l} |D^r f;\ L_{p,\alpha}(\Omega)| =$$

$$= |f; L_p(g)| + |f; L^l_{p,\alpha}(\Omega)| \quad (|f; W^0_{p,\alpha}(\Omega, g)| = |f; L_{p,\alpha}(\Omega)|),$$

$$|f; W^l_{p,\alpha,\alpha_0}(\Omega)| = |f; L_{p,\alpha_0}(\Omega)| + |f; L^l_{p,\alpha}(\Omega)|,$$

where $D^r f$ is a generalized derivative of order

$$|r| = \sum_{i=1}^{n} r_i.$$

Denote by $\overset{\circ}{B}$ the closure in the norm of the class B of functions from $C^\infty_0(\Omega)$. By definition, a domain Ω with locally Lipschitz boundary belongs to the class Z_m if

$$\Omega = G + \sum_{j=1}^{N} \Omega_j,$$

where G is a finite domain, $N < \infty$, and the domain Ω_j $(j = 1, \dots, N)$ has the following properties after a nonsingular linear transformation, its own for each j:

1) for $X \in \Omega_j$, the coordinates x_i have $|x_i| \le c < \infty$ $(i = m+1, \dots, n)$, while the remaining coordinates $x_1, \dots, x_m$ are unbounded;
2) if one passes from the orthogonal coordinates X to cylindrical coordinates

$$Y: \quad y_1 = |X_m| = \left(\sum_{i=1}^{m} x_i^2\right)^{1/2}, \quad y_2 = \varphi, \quad y_3 = \theta_1, \dots, y_m = \theta_{m-2}, \quad y_{m+1} = x_{m+1}, \dots$$

$$\dots, y_n = x_n \quad (y_1 = x_1 \text{ for } m = 1),$$

then for some $i > 1$, depending on j, the domain Ω_j is written in the form

$$\Omega_j = \{Y:\ a^{(j)}_k < y_k < b^{(j)}_k\ (k \ne i), \quad a^{(j)}_i < y_i < \psi^{(j)}_i(P_i)\ (\psi^{(j)}_i(P_i) \ge b^{(j)}_i)\},$$

where $a^{(j)}_k, b^{(j)}_k$ are constants, $|a^{(j)}_k| < \infty$ $(k = 1, \dots, n)$, $b^{(j)}_1 = \infty$, $|b^{(j)}_k| < \infty$ $(k = 2, \dots, n)$, $P_i = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$, and $\psi^{(j)}_i(P_i)$ is a single-valued continuous bounded function of P_i.

Theorem 1. Let $f \in W^l_{p,\alpha}(\Omega, g)$, $\Omega \in Z_m$, $1 \le m \le n$; let α be a real number; $1 \le p \le \infty$; l, k natural numbers, $0 \le k < l$; $\beta = \max(\alpha, m/p - 1 + \varepsilon)$, $\varepsilon > 0$; $\alpha_k = \beta + l - k$; $|r| = k$.

Then

$$|D^r f;\, L_{p,\alpha_k}(\Omega)| \ll |f;\, W^l_{p,\alpha}(\Omega, g)|^*.$$

Corollary 1. Let $\Omega \in Z_m$, $1 \le m \le n$; let α be a real number; $1 \le p \le \infty$; l a natural number;

$$\beta = \max(\alpha, m/p - 1 + \varepsilon), \quad \varepsilon > 0; \quad \alpha_0 = \beta + l.$$

Then the classes $W^l_{p,\alpha}(\Omega, g)$, $W^l_{p,\alpha,\alpha_0}(\Omega)$ coincide up to equivalence of norms.

Theorem 1 for $\Omega = E_n$ is contained in the works $(^{3-5})$.

Theorem 2. Let $\Omega \in Z_m$, $1 \le m \le n$; let α be a real number; $1 \le p \le \infty$; l, k natural numbers, $0 \le k < l$; $\beta = \max(\alpha, m/p - 1 + \varepsilon)$, $\varepsilon > 0$; $\alpha_k = \beta + l - k$.

Then $W^l_{p,\alpha}(\Omega, g)$ is embedded in $W^k_{p,\alpha_k+\varepsilon}(\Omega, g)$ completely continuously.

Theorem 3. Let $\alpha \ge 0$; $1 \le p \le \infty$; l, k be natural numbers, $0 \le k < l$; $\varepsilon > 0$; $\alpha_k = \alpha + l - k$.

Then the embedding $\mathring{W}^l_{p,\alpha}(\Omega, g)$, $\mathring{W}^k_{p,\alpha_k+\varepsilon}(\Omega, g)$ is completely continuous.

Corollary. Let the unbounded domain Ω contain no sphere of positive radius; $\alpha \ge 0$; $1 \le p < \infty$; l, k be natural numbers, $0 \le k < l$; $\varepsilon > 0$; $\alpha_k = \alpha + l - k$.

Then the embedding $\mathring{L}^l_{p,\alpha}(\Omega)$ in $\mathring{L}^k_{p,\alpha_k+\varepsilon}(\Omega)$ is completely continuous.

The stated theorems are sharp in the sense that the exponents of the weight cannot be decreased by $\varepsilon > 0$.

Theorem 4. Let the unbounded domain Ω contain no sphere of positive radius. Suppose, further, that $\alpha \ge 0$; $\varepsilon > 0$; $\alpha_0 = \alpha + l + \varepsilon$; $\beta = \alpha$ for $\alpha > n/2 - 1$, $\beta = \alpha + \varepsilon$ for $\alpha \le n/2 - 1$; $L(X, D)$ is a linear elliptic self-adjoint operator of order $2l$ (D is the differentiation operator in the sense of distributions)

$$L(X, D) = (-1)^l \sum_{\sum \alpha_i = l} \frac{l!}{\alpha_1! \cdots \alpha_n!} \frac{\partial^l}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \left(a_{\alpha_1 \ldots \alpha_n}(X) \frac{\partial^l}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right) + b(X),$$

$$\sum_{\sum \alpha_i = l} \frac{l!}{\alpha_1! \cdots \alpha_n!} a_{\alpha_1 \ldots \alpha_n}(X) (\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n})^2 \ge$$

$$\ge \frac{B}{(1 + |X|^2)^\alpha} \sum_{\sum \alpha_i = l} \frac{l!}{\alpha_1! \cdots \alpha_n!} (\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n})^2 \quad \text{for } X \in \Omega,\ \vec{\xi} \in E_n,$$

$$a(X, D) = c^{-1}(X) L(X, D);$$

$a_{\alpha_1\dots\alpha_n}(X)$, $b(X)$, $c(X)$ are infinitely differentiable coefficients in Ω;

$$|a_{\alpha_1\dots\alpha_n}(X)| \le A/(1+|X|^2)^{\alpha}; \quad 0 \le$$

$$\le b(X) \le A/(1+|X|^2)^{\beta+l}; \qquad 0 < c(X) \le A/(1+|X|^2)^{\alpha_0};$$

$A,\ B$ are positive constants.

Define the operator T in $L_{2,c(X)}(\Omega)$ as follows:

$$D(T) = \overset{\circ}{L}{}^{l}_{2,\alpha}(\Omega) \cap \{f \in L_{2,c(X)}(\Omega) : a(X,D)f \in L_{2,c(X)}(\Omega)\},$$

$$Tf = a(X,D)f, \qquad f \in D(T).$$

Then T is a closed linear operator; the spectrum $\sigma(T)$ is discrete, has no finite limit points, and is situated in the half-plane $\operatorname{Re}\lambda > 0$; for $\lambda \notin \sigma(T)$ the resolvent

$$R_\lambda(T) = (\lambda I - T)^{-1}$$

is completely continuous in $L_{2,c(X)}(\Omega)$.

We note that the operator $L(X,D)$ is equivalent to the operator $a(X,D)$ in the sense that the equations $L(X,D)u = f$ for $f \in L_{2,c^{-1}(X)}(\Omega)$ and

$$a(X,D)u = c^{-1}(X)f = g$$

for $g \in L_{2,c(X)}(\Omega)$ are equivalent (a generalized solution of one is a generalized solution of the other).

If, in the definition of the spectrum of the operator A, one replaces the resolvent

$$R_\lambda(A) = (\lambda I - A)^{-1}$$

by the resolvent

$$R_{\lambda,c}(A) = (\lambda c(X) I - A)^{-1},$$

then we obtain the definition of the c-spectrum of the operator A.

Corollary. Let the hypotheses of Theorem 4 be satisfied. Define the operator T_1 as follows:

$$D(T_1) = \overset{\circ}{L}{}^{l}_{2,\alpha} \cap \{f \in L_{2,c(X)}(\Omega) : L(X,D)f \in L_{2,c^{-1}(X)}(\Omega)\},$$

$$T_1 f = L(X,D)f, \qquad f \in D(T).$$

* By the expression $A \ll B$ here and below is meant $A \le cB$, where the positive constant c does not depend

Then T_1 is a closed linear operator, the c-spectrum $\sigma_c(T_1)$ is discrete, has no finite limit points, and is situated in the half-plane $\operatorname{Re}\lambda > 0$; for $\lambda \notin \sigma_c(T_1)$ the resolvent

$$R_{\lambda,c}(T_1) = (\lambda c(X)I - T_1)^{-1}$$

is completely continuous from $L_{2,c^{-1}(X)}(\Omega)$ into $L_{2,c(X)}(\Omega)$.

For $n = 1$ and $l = 1$, the c-spectrum of the operator $L(X, D)$, under other restrictions on the coefficients, was first considered by K. Friedrichs (^{6}).

Theorem 5. Let the hypotheses of Theorem 4 be satisfied, with the sole exception that

$$\alpha_0 = \beta + l.$$

Then the equation

$$L(X, D)u = f$$

for a function f such that

$$(1 + |X|)^{\alpha_0} f \in L_2(\Omega),$$

has, moreover, a unique generalized solution

$$u \in \overset{\circ}{L}{}^{l}_{2,\alpha}(\Omega),$$

and the inequality

$$|u;\ \overset{\circ}{L}{}^{l}_{2,\alpha}(\Omega)| \le |(1 + |X|)^{\alpha_0} f;\ L_2(\Omega)|$$

holds.

The operator $L(X, D)$ for $l = 1$ was considered earlier by L. D. Kudryavtsev ($^{7-9}$) in connection with the variational method for solving elliptic equations posed in a half-space.

In conclusion, the author expresses his gratitude to his scientific supervisor Prof. L. D. Kudryavtsev for posing the problem and for his attention to the work.

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Received
20 VI 1968

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Note: Figure translations are in progress. See original paper for figures.