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A BICOMPACTUM WITH NONCOINCIDING INDUCTIVE DIMENSIONS

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P. S. Aleksandrov posed the question of the coincidence of the dimensional invariants ind and Ind for bicompacta. The present note is devoted to the construction of a bicompactum X with $\operatorname{ind} X = 2$, $\operatorname{Ind} X = 3$.

Let Y_1 be the set of sequences $\{x_\alpha\}_{\alpha<\omega_1}$, enumerated by transfinite numbers (strictly) less than ω_1, of points of the segment $I = [0, 1]$, such that if at some place in the sequence there stands 0 or 1, then in all subsequent places the same number stands. The set Y_1 carries the natural lexicographic order, i.e., $\{y_\alpha\} < \{x_\alpha\}$ if $y_\beta < x_\beta$, where β is the least of the transfinite numbers at which the sequences do not coincide. It is not difficult to verify that the space Y_1 in the order topology is a connected bicompactum. We may represent the space Y_1 as the union of two everywhere dense subsets Y_1^0 and Y_1^1, where Y_1^1 consists of points of countable character (those sequences in which, beginning with some place, there stands 0 or 1), and Y_1^1 consists of points of uncountable ($\aleph_1$) character, which are not limit points for any countable set (those sequences in which there are no 0 and 1). The family of all intervals of the form (r_1, r_2) and half-intervals of the form $[\min Y_1, r_2)$ and $(r_1, \max Y_1]$, where $r_1, r_2 \in Y_1^0$, forms a base of the space Y_1. Let Ξ be the family of those ordered pairs of elements of this base in which the second member lies with its closure in the first and both are simultaneously either intervals or half-intervals. The cardinality of the family Ξ is equal to the cardinality of the set Y_1^0, which is equal to $c^{\aleph_0} \cdot \aleph_1 = c$. The intersection of Y_1^1 with any open set has cardinality $c^{\aleph_1} \geq c$. By a theorem of Sierpiński (*1*), the subspace Y_1^1 can be represented as the union of $\geq c$ subsets everywhere dense in Y_1^1 and, consequently, everywhere dense in Y_1. Let R, P_α^ξ ($\xi \in \Xi, \alpha < \omega_1$) be pairwise disjoint everywhere dense subsets of the set Y_1^1, the family $\{P_\alpha^\xi\}$ of such subsets being indexed by transfinite numbers less than ω_1 and by elements of the family Ξ. Similarly, from Y_1^0 we can single out a family $\{Q_\alpha^\xi\}_{\xi\in\Xi,\alpha<\omega_1}$ of pairwise disjoint sets everywhere dense in Y_1.

Fix $\alpha < \omega_1$. To an element ξ of the family Ξ we put in correspondence the set b_ξ^α, which is an interval (y_1, y_2) if ξ consists of intervals, and a half-interval $[\min Y_1, y_2)$ or $(y_1, \max Y_1]$ if ξ consists of half-intervals, the set b_ξ^α being chosen in such a way that it lies in the first member of the pair ξ and contains the second, and $y_1, y_2 \in P_\alpha^\xi$. The resulting family $B_\alpha = \{b\}_{\xi \in \Xi}$ is, as is easy to see, a base of the space Y_1.

Thus, we have obtained $\aleph_1$ bases B_α, $\alpha < \omega_1$, of the space Y_1. For us it will be essential that no two elements of the family $\bigcup_{\alpha<\omega_1} B_\alpha$ have common open ends (this follows from the fact that the sets P_α^ξ are pairwise disjoint).

Take some point $\vartheta \in Y_1^1 \setminus R$. Fix a sequence

the family $\{H_\alpha\}_{\alpha<\omega_1}$, indexed by transfinite numbers less than ω_1, containing the point ϑ, of intervals contracting to ϑ such that $[H_\alpha] \subset H_\beta$ if $\alpha > \beta$. To an element $b_\xi^{(\alpha)}$ of the base B_α we put in correspondence an interval $c_\xi^{(\alpha)}$ with ends from P_α^ξ such that

$$[H_{\alpha+1}] \subset c_\xi^{(\alpha)} \subset [c_\xi^{(\alpha)}] \subset H_\alpha.$$

As is easy to see, the family D_1 of sets of the form $b_\xi^{(\alpha)} \times c_\xi^{(\alpha)}$ forms a base at the points of the set $\{\vartheta\} \times Y_1$ of the space $Y_2 = Y_1 \times Y_1$. We note (this is important) that the intersection of the boundaries of two squares* of the family D_1 is either empty, or consists of a finite number of points which are not vertices of squares of the family D_1. To an element $d = b_\xi^{(\alpha)} \times c_\xi^{(\alpha)}$ of the family D_1 we put in correspondence (everywhere dense on the boundary) a set $s(d)$ of boundary points at which at least one (and, as is easy to see, no more than one) coordinate belongs to Q_α^ξ. If a vertical or horizontal line intersects the boundary of a square of the family D_1 in a whole interval, then it contains no more than two points of the set $S_1 = \bigcup_{d\in D_1} s(d)$. Let S_2 be the set of those vertices of squares of the family D_1 neither of whose coordinates is equal to $\min Y_1$ or $\max Y_1$. Let S_3 consist of the points of the set $\{\vartheta\} \times Y_1$ which do not lie on the boundaries of squares of the family D_1.

Let D_2^* be the family of pairwise products of sets of the form (r_1, r_2), or $[\min Y_1, r_2)$, or $(r_1, \max Y_1]$, where $r_1, r_2 \in R$. Let D_2 be that part of D_2^* consisting of sets not intersecting $\{\vartheta\} \times Y_1$. As is easy to see, the family D_2 forms a base at the points of $Y_2 \setminus (\{\vartheta\} \times Y_1)$.

Let $R^2 = R \times R$.

A sequence $\{\delta_\alpha\}_{\alpha\leq\omega_1}$, indexed by transfinite numbers not exceeding ω_1, of pairwise disjoint intervals lying in Y_1, will be called marked if: a) δ_α consists of one point if α is a limit transfinite number, and of more than one point if α is an isolated transfinite number; b) the sequence $\{\min \delta_\alpha\}$ is either decreasing or increasing; c) the set $\bigcup_{\alpha\leq\omega_1} \delta_\alpha$ is closed in Y_1. We assign to the family T a bicompactum t lying in Y_2 if for it there is found a pair $\{\delta_\alpha^1\}$ and $\{\delta_\alpha^2\}$ of marked sequences (we shall then put one such pair in correspondence with the bicompactum t) such that $t = \bigcup_{\alpha\leq\omega_1} t_\alpha$, where $t_\alpha = t \cap (\delta_\alpha^1 \times \delta_\alpha^2) \neq \emptyset$ and $\operatorname{ind} t_\alpha = 1$ if

α is an isolated transfinite number, and the single point from $\delta^1_{\omega_1} \times \delta^2_{\omega_1} = t_{\omega_1} \subset t$ does not belong to R^2.

Fix a decomposition of the set of all isolated transfinite numbers less than ω_1 into two sets σ_1 and σ_2, cofinal in ω_1.

There exists a mapping $\lambda : K \to I$ of a Cantor perfect set onto an interval, which identifies pairwise the endpoints of adjacent intervals, takes them to all dyadic-rational points of the interval (except 0 and 1), and preserves order. We split the dyadic-rational points into two everywhere dense sets Q_1 and Q_2. Let λ_i $(i = 1, 2)$ be a mapping of the Cantor perfect set onto itself, identifying those points which are identified under λ and pass into Q_i.

Let Z be the space of transfinite numbers not exceeding some transfinite number α. Between the transfinite numbers β and $\beta + 1 \leq \alpha$ we insert (without making any identifications!) a copy K_β of the Cantor perfect set. The order in the resulting set C_α is determined by the order on Z and on the Cantor perfect set and by the convention that all points of K_β lie between β and $\beta + 1$. The space C_α, taken in the order topolo-

* By a square we shall mean a set of the form $J_1 \times J_2$, where J_i $(i = 1, 2)$ is an interval, or a half-open interval, or a segment. By the boundary is meant the topological boundary with respect to Y_2. A segment (horizontal or vertical, respectively) is a set of the form $J \times \{y\}$ or $\{y\} \times J$, where J is a (nontrivial) segment in Y_1, $y \in Y_1$. If $J = Y_1$, then the corresponding segment is called horizontal or vertical.

is a bicompactum. The points of the set $Z \subset C_\alpha$ will be called transfinites.

Let χ be a transfinite whose character is greater than the cardinality of the set of all subsets of the space Y_2. To each transfinite smaller than χ we assign either some bicompactum of the family T, or a point of the set $Y_2 \setminus (R^2 \cup S_1)$, in such a way that the set of those transfinites to which a certain fixed point or a certain fixed bicompactum has been assigned is cofinal in χ.

In the product $Y_2 \times C_\chi$ we make the following identifications:

a) if to a transfinite $\beta < \chi$ there is assigned the bicompactum

$$t = \bigcup_{\alpha \leqslant \omega_1} t_\alpha$$

of the family T, then in the set $\{y\} \times K_\beta$ we perform the identification λ, if $y \in t_\alpha$, where α is a limit transfinite, and the identification λ_i, if $y \in t_\alpha$, where $\alpha \in \sigma_i$;

b) if to a transfinite $\beta < \chi$ there is assigned a point $y_0 \in Y_2 \setminus (S_2 \cup S_3)$, then in the set $\{y_0\} \times K_\beta$ we perform the identification λ, in the set $\{y\} \times K_\beta$ we perform the identification λ_1, if y belongs to the vertical line passing through y_0, and the identification λ_2, if y belongs to the horizontal line passing through y_0;

c) if to a transfinite $\beta < \chi$ there is assigned a point $y_0 \in S_2$, which is a vertex of a square $d \in D_1$, then in the set $\{y_0\} \times K_\beta$ we perform the identification λ, in the set $\{y\} \times K_\beta$, where y belongs to the horizontal or vertical line passing through the point y_0, we perform the identification λ_1, if the segment joining y to y_0 intersects the boundary of the square in some (nontrivial) segment, and the identification λ_2 otherwise;

d) if to a transfinite $\beta < \chi$ there is assigned a point $y_0 \in S_3$, then in the set $\{y_0\} \times K_\beta$ we perform the identification λ; in the set $\{y\} \times K_\beta$, where y lies on the horizontal line passing through the point y_0, we perform the identification λ_1, if y lies to the left of y_0, and the identification λ_2, if y lies to the right of y_0.

As a result of the factorization Φ carried out, we obtain a certain (Hausdorff) bicompactum X, with

$$\operatorname{ind} X = 2, \qquad \operatorname{Ind} X = 3, \qquad X = X_1 \cup X_2,$$

where

$$X_1 = \Phi([\min Y_1, \vartheta] \times Y_1 \times C_\chi), \qquad X_2 = \Phi([\vartheta, \min Y_1] \times Y_1 \times C_\chi),$$
$$\operatorname{Ind} X_1 = \operatorname{Ind} X_2 = 2.$$

We give an outline of the verification of the last assertions. First of all note that all dimensions of the spaces X, X_1, X_2 are not less than 2 and not greater than 3.

Let V be a set open in Y_2, and let

$$(V) = \Phi(V \times C_\chi).$$

Note that if $V \in D_1 \cup D_2$, then the boundary of the set (V) is one-dimensional. Let U be a set open in C_χ; the (open) set $C_\lambda(U)$ consists of those points of the space X whose full inverse image under Φ lies in $Y_2 \times U$. As is easy to see, the sets of the form $(V) \cap C_\lambda(U)$ form a base of the space X. At any point $x = \Phi((y, c))$ there is an arbitrarily small neighborhood of this form with $V \in D_1 \cup D_2$ such that, if c is a transfinite or the point (y, c) belongs to a set $\{y\} \times K_\beta$ in which the identification λ has not been performed, then the boundary of the neighborhood lies entirely in the boundary of (V), or, if the point (y, c) belongs to a set $\{y\} \times K_\beta$ in which the identification λ has been performed, then the boundary of the neighborhood decomposes into the union of two closed one-dimensional sets F_1 and F_2, where F_1 lies in the boundary of (V), F_2 lies in the boundary of $C_\lambda(U)$, and either $F_1 \cap F_2 = \emptyset$, or F_1 consists of a finite number of (nonintersecting) segments. In all these cases the boundary of the neighborhood is one-dimensional and, consequently, $\operatorname{ind} X = 2$. Every closed set in X_i ($i = 1$ or 2) has an arbitrarily tight neighborhood with one-dimensional boundary, which is the union of a finite number of sets of the form $(V) \cap C_\lambda(U) \cap X_i$, where $V \in D_2^*$; therefore

$$\operatorname{Ind} X_1 = \operatorname{Ind} X_2 = 2.$$

Lemma 1. Let

$$\Gamma_1 = Y_1 \times \{\min Y_1\}, \qquad \Gamma_2 = Y_1 \times \{\max Y_1\}.$$

Let the set F separate Γ_1 and Γ_2. Then F contains either a) a bicompactum of the fam–

of T, or b) a pair of intersecting segments, one of which lies on the boundary of some square of the family D_1, while the other lies outside this boundary, or c) a horizontal segment not lying on the boundary of a square of the family D_1, intersecting (passing from one side to the other) the segment $\{\vartheta\} \times Y_1$, or d) a segment of the form $\{\vartheta\} \times J$, or $J \times \{y\}$, or $\{y\} \times J$, where $y \in Y_1$ is a point of countable character.

Lemma 2. Let G_1 and G_2 be two open disjoint sets in X; $F = X \setminus (G_1 \cup G_2)$ consists of points limiting both for G_1 and for G_2. Then either F contains some set open in $\Phi(Y_1 \times \{\chi\})$, or there exists a transfinite $\alpha < \chi$ such that, as soon as $\Phi((y, \chi)) \in F$ and $\alpha \leq c \leq \chi$, then $\Phi((y, c)) \in F$.

From Lemmas 1 and 2 and the construction of the space X it follows easily that

$$\operatorname{Ind} X = 3.$$

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References

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Note: Figure translations are in progress. See original paper for figures.