ON INTEGRAL CONSERVATION LAWS

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Abstract

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MECHANICS

V. A. ZHELNOROVICH

ON INTEGRAL CONSERVATION LAWS

FOR CONTINUOUS MEDIA IN THE GENERAL THEORY OF RELATIVITY

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The formulation of integral conservation laws for the energy, momenta, and moments of the quantity of motion in the general theory of relativity is connected with difficulties arising in generally accepted theories (^{1,2}), owing to the possibility of introducing various energy-momentum complexes, and also owing to the indeterminacy of integration in a Riemannian space of events. Below a new approach is proposed for obtaining integral conservation laws on the basis of a variational principle, connected with the introduction of uniquely determined canonical equations of state (^{3,6}).

1. Canonical equations of state. Let x^i be the variables of the observer' s coordinate system in a pseudo-Riemannian space of events V of index 3. Introduce in the space V a nonholonomic coordinate system $\tilde{x}^\alpha$, consisting of orthonormal frames (tetrads) assigned to each point of the space V and determined by the Lamé metric matrix h_i^α.

We also introduce in the space V a coordinate system with variables ξ^p, accompanying the material continuous medium considered in V. The law of motion of the medium is determined by the functions $x^i = x^i(\xi^p)$ or by the functions $\xi^p = \xi^p(x^i)$.

We shall use the variational principle in the form proposed by L. I. Sedov ($^{4-6}$):

$$\delta \int_{\dot{V}4} \Lambda \, d\tau + \delta W^* + \delta W = 0.$$

For details on the meaning of the variational equality see the works (^{4,6}). Consider the case when the Lagrangian Λ is defined as a function of the following system of arguments:

$$\rho,\ \mu^A,\ \mu^A_{,i},\ h_i^\alpha,\ h^\alpha_{i,j},\ h^\alpha_{i,jk},$$

where ρ is the density of the material continuous medium; μ^A are field functions, which may be independent components of tensors or spinors ($^{7-9}$), given in the nonholonomic system $\tilde{x}^\alpha$. The index after a comma denotes differentiation with respect to the variables x^i: $\mu^A_{,i} = \partial\mu^A/\partial x^i$. For simplicity we set $\delta W^* = 0$.

Using ordinary methods (^{3,4}), from the variational principle we find the functional δW

$$\delta W = \int_\Sigma \left\{ \left[\left(-\rho^2 \frac{\partial \Lambda/\rho}{\partial \rho} - \Lambda \right) u_i u^k + \rho^2 \frac{\partial \Lambda/\rho}{\partial \rho} \right] \delta x^i - \frac{\partial \Lambda}{\partial \mu^A_{,k}} \delta \mu^A - \right.$$

$$\left. - \left(\frac{\partial \Lambda}{\partial h^\alpha_{i,k}} - \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^j} \frac{\partial \Lambda \sqrt{-g}}{\partial h^\alpha_{i,jk}} \right) \partial h^\alpha_i - \frac{\partial \Lambda}{\partial h^\alpha_{i,jk}} \partial h^\alpha_{i,j} \right\} n_k \, \varepsilon \, d\sigma_3. \qquad (1)$$

Here u^k are the contravariant components of the unit velocity vector of the points of the medium; n_k are the covariant components of the unit (imaginary-unit-

···vector of the external normal to the surface Σ bounding the volume V_4; $d\sigma$ is the invariant element of the surface Σ; $g = \det\|g_{ij}\|$; g_{ij} are the covariant components of the metric tensor of event space.

Suppose now that the piecewise-smooth surface Σ admits a parametric representation $x^i = f^i(u^1 u^2 u^3)$. Denote $\zeta^i_\sigma = \partial f^i/\partial u^\sigma$. Then the components $\partial h^\alpha_{i,j} = \partial h^\alpha_i/\partial x^j$ can be represented in the form $^{(3)}$

$$\partial h^\alpha_{i,j} = \varepsilon n_j \frac{\partial}{\partial n} \partial h^\alpha_i + \zeta^\varepsilon_j \frac{\partial}{\partial u^\varepsilon} \partial h^\alpha_i. \qquad (2)$$

Here $\zeta^\varepsilon_j = G^{\sigma\varepsilon} g_{ij} \zeta^i_\sigma$, $\varepsilon = n^p n_p$, and $G^{\sigma\varepsilon}$ are the contravariant components of the first metric tensor of the surface Σ.

Replacing in (1) the quantities $\partial h^\alpha_{i,j}$ by formula (2), we represent the functional δW in the following canonical form:

$$\delta W = \int_\Sigma \left\{ n_k \left[\left(-\rho^2 \frac{\partial \Lambda/\rho}{\partial \rho} - \Lambda \right) u_i u^k + \rho^2 \frac{\partial \Lambda/\rho}{\partial \rho} \delta^k_i \right] \delta x^i - n_k \frac{\partial \Lambda}{\partial \mu^A_{,k}} \partial \mu^A - \right.$$

$$\left. - \theta^i_\alpha \partial h^\alpha_i - \varepsilon n_k n_j \frac{\partial \Lambda}{\partial h^\alpha_{i,jk}} \frac{\partial \, \partial h^\alpha_i}{\partial n} \right\} d\sigma_3 - \int_\Sigma \frac{1}{\sqrt{-G}} \frac{\partial}{\partial u^\varepsilon} \left(\sqrt{-G} \, \zeta^\varepsilon_j n_k \frac{\partial \Lambda}{\partial h^\alpha_{i,jk}} \partial h^\alpha_i \right) d\sigma_3.$$

Here $G = \det\|G_{\varepsilon\sigma}\|$,

$$\theta^i_\alpha = n_k \left(\frac{\partial \Lambda}{\partial h^\alpha_{i,k}} - \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^j} \frac{\partial \Lambda \sqrt{-g}}{\partial h^\alpha_{i,jk}} \right) - \frac{1}{\sqrt{-G}} \frac{\partial}{\partial u^\varepsilon} \left(\sqrt{-G} \zeta^\varepsilon_j n_k \frac{\partial \Lambda}{\partial h^\alpha_{i,jk}} \right). \quad (3)$$

The form (3) of the functional δW, in contrast to the form (1), is determined uniquely $^{(3)}$.

2. Integral conservation laws

Let us define the change of the action integral $\delta_G \int \Lambda \, d\tau$ under an arbitrary transformation of the variables x^i of the observer' s coordinate system $x^{i'} = \varphi^i(x^p)$. To do this, it suffices to set, in the expression for the arbitrary variation of the action integral, the variations ∂x^i, $\partial \mu^A$, ∂h^α_i equal to

$$\partial x^i = \delta \eta^i, \qquad \partial \mu^A = -\mu^A_{,i} \delta \eta^i, \qquad \partial h^\alpha_i = -\delta \eta^j h^\alpha_{i,j} - h^\alpha_j (\delta \eta^j)_{,i}, \qquad (4)$$

where $\delta \eta^i = \varphi^i(x^p) - x^i$ (to within first-order infinitesimals). By the Euler equations we have $\delta \int \Lambda \, d\tau = -\delta W$. Substituting the values of the parameter variations defined by equalities (4) into expression (3) for the functional δW, we obtain the magnitude of the change of the action integral due to the transformation of the variables x^p, under the condition $\partial \delta \eta^i / \partial x^j = 0$,

$$\delta_G \int_{V_4} \Lambda \, d\tau = -\int_\Sigma P_i \delta \eta^i \, d\sigma_3 + \int_\Sigma \frac{1}{\sqrt{-G}} \frac{\partial}{\partial u^\varepsilon} \left(P^\varepsilon_i \sqrt{-G} \, \delta \eta^i \right) d\sigma_3, \qquad (5)$$

where

$$P_i = \varepsilon \left[\left(-\rho^2 \frac{\partial \Lambda / \rho}{\partial \rho} - \Lambda \right) u_i u^k + \rho^2 \frac{\partial \Lambda / \rho}{\partial \rho} \delta^k_i + \frac{\partial \Lambda}{\partial \mu^A_{,k}} \mu^A_{,i} + \varepsilon n_j \frac{\partial \Lambda}{\partial h^\alpha_{l,jk}} \frac{\partial}{\partial n} h^\alpha_{l,i} \right] n_k + \varepsilon \theta^l_\alpha h^\alpha_{l,i}, \qquad P^\varepsilon_i = -\varepsilon \zeta^\varepsilon_j n_k \frac{\partial \Lambda}{\partial h^\alpha_{l,j}} \quad (6)$$

If the action integral is invariant with respect to the group of general coordinate transformations, then $\delta_G \int \Lambda \, d\tau$, determined by formula (5), becomes zero by virtue of the Euler equations. Under this assumption, from (5) there follows the integral conservation law for energy-momentum

$$\int_\Sigma P_i \, d\sigma_3 = \int_\Sigma \frac{1}{\sqrt{-G}} \frac{\partial}{\partial u^\varepsilon} \left(P^\varepsilon_i \sqrt{-G} \right) d\sigma_3. \qquad (7)$$

Let us now take as the surface of integration in equations (7) the three-dimensional surface Σ^*, consisting of the three-dimensional volumes V_3 at the instants $x^4 = ct_1$, $x^4 = ct_2$ (c is the speed of light in vacuum) and the coordinate surface Σ_c, resting on the two-dimensional surfaces Σ_2 that bound the volumes V_3 at $x^4 = ct_1$, $x^4 = ct_2$ (Fig. 1). It is obvious that the surfaces Σ_2 at the instants ct_1, ct_2 are the edges of the surface Σ^*.

As the parameters u^σ of the surface Σ_c we choose the parameters u^1, u^2, t, where u^1, u^2 are the parameters of the surface Σ_2.

Fig. 1

Next divide equation (7) by $t_2 - t_1$ and pass in it to the limit as $t_2 \to t_1$. As a result we obtain the integral relation

$$\frac{\partial}{\partial t} S_i + \frac{1}{c} \int_{\Sigma_2} (P_i \sqrt{-G})_c \, du^1 \, du^2 = 0, \tag{8}$$

where

$$S_i = \frac{1}{c} \int_{V_3} (P_i)_V \, dV_3 - \frac{1}{c} \int_{\Sigma_2} [(P_i^\varepsilon)_V n_\varepsilon - (P_i^\varepsilon)_c \nu_\varepsilon] \, d\sigma_2. \tag{9}$$

In formulas (8), (9), dV_3 denotes an element of the volume V_3; n_ε are the components of the unit normal vector to the surface Σ_2, lying in V_3; ν_ε are the components of the unit vector of the exterior normal to Σ_2 for $x_4 = ct_2$, lying on the surface Σ_c. The expressions $(P_i)_c$, $(P_i^\varepsilon)_c$ mean that the components n_k of the normal vector entering into P_i, P_i^ε are components of the vector of the exterior normal to the surface Σ_c. The expressions $(P_i)_V$, $(P_i^\varepsilon)_V$ mean that the components n_k entering into P_i, P_i^ε are components of the vector of the exterior normal to the surface $x^4 = ct_2$.

Using the Ostrogradsky-Gauss theorem for the three-dimensional volume V_3, taking into account definitions (6), (9), the expression for the components S_i may also be rewritten in the form

$$S_i = \frac{1}{c} \int_{V_3} (\theta_i^k n_k)_V \, dV_3 + \frac{1}{c} \int_{\Sigma_2} (P_i^\varepsilon \nu_\varepsilon) \, d\sigma_2, \tag{10}$$

where θ_i^k are the components of the energy-momentum pseudotensor satisfying the differential conservation law $\partial(\sqrt{-g}\theta_i^k)/\partial x^k = 0$,

$$\theta_i^k = \left(-\rho^2 \frac{\partial \Lambda/\rho}{\partial \rho} - \Lambda \right) u_i u^k + \rho^2 \frac{\partial \Lambda/\rho}{\partial \rho} \delta_i^k + \frac{\partial \Lambda}{\partial \mu^A_{,k}} \mu^A_{,i} +$$
$$+ \left(\frac{\partial \Lambda}{\partial h^a_{l,k}} - \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^j} \frac{\partial \Lambda \sqrt{-g}}{\partial h^a_{l,jk}} \right) h^a_{l,i} + \frac{\partial \Lambda}{\partial h^a_{l,jk}} h^a_{l,ij}. \tag{11}$$

In the case where Λ depends only on the first derivatives $h^a_{i,k}$, definition (10) coincides with the usual one.

The advantage of definition (10) is its independence of the arbitrariness in the definition of the integrands in the functional (1) (and, consequently, its independence of the arbitrariness in the definition of the energy-momentum pseudotensors), and the continuity of the stresses $(P_i)_c$ on the discontinuity surfaces (3).

Let us note that a feature of definition (10) is the presence in (10) of an integral over the surface Σ_2, which in general cannot be reduced to an integral over the volume V_3.

If, as the Lagrangian for the gravitational field, one takes the curvature of the space of events $-\dfrac{1}{2\chi}R$ (χ is Einstein' s gravitational

constant), the surface part of the components S_i corresponding to the term $-\dfrac{1}{2\varkappa}R$ can be written in the form

$$S_i = -\frac{1}{2\varkappa c}\int_{\Sigma_2} g_{jk,i}\left(n^k \xi^j_e v^e\right)_c d\sigma_2.$$

The method set out for obtaining integral conservation laws for energy-momentum is also applicable to deriving equations for the moments of momentum S_{ij}, and carries over without change to the case when Λ depends on higher derivatives of the defining parameters. It then turns out that the components S_i, S_{ij} are represented by integrals over the surface Σ_2 and, generally speaking, depend on the curvature of the surface Σ_2.*

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Moscow State University
named after M. V. Lomonosov

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* The corresponding conclusions within the framework of the special theory of relativity were reported at the Third All-Union Congress on Theoretical and Applied Mechanics. We note that, in magnetic media, the existence of surface internal mechanical moments must lead to the existence of surface internal electromagnetic moments. The effects described can be represented mathematically either by considering a Lagrangian Λ that depends on higher derivatives, or by introducing into the action integral S the surface integral

$$S = \int_{V_4} \Delta \, d\tau + \int_{\Sigma + S_\pm} \Phi \, d\sigma.$$

Note: Figure translations are in progress. See original paper for figures.

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