ON THE INDECOMPOSABILITY OF TRANSITIVE SUBGROUPS OF THE GROUP \(SF(X)\)

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Abstract

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ON THE INDECOMPOSABILITY OF TRANSITIVE SUBGROUPS OF THE GROUP $SF(X)$

1. Let X be an arbitrary nonempty set; $S(X)$ the symmetric group acting on X, i.e., the group of all bijective mappings $f : X \to X$. For infinite X we introduce the group $SF(X)$, consisting of all such $f \in S(X)$ that $f(x) \neq x$ only for a finite set of points $x \in X$. Obviously, $SF(X)$ is a transitive locally finite invariant subgroup of the group $S(X)$. If $\mathfrak{A}$ is a group, then an arbitrary homomorphism $g : \mathfrak{A} \to S(X)$ is called a representation of the group $\mathfrak{A}$ by substitutions of $S(X)$. A representation $h : \mathfrak{A} \to S(Y)$ is called equivalent to the representation g if there exists a bijective mapping $\varphi : X \to Y$ such that for every $a \in \mathfrak{A}$ the equality $h(a)\varphi = \varphi g(a)$ holds.

A group Γ from $S(X)$ is called regular if it is transitive and its stationary subgroup coincides with the identity subgroup. As is known, every regular subgroup of $S(X)$ is the image of the left regular representation of some group. Consequently, if X is an infinite set and Γ is a regular subgroup of $S(X)$, then $\Gamma \cap SF(X) = (e)$, where e is the identity of $S(X)$.

We shall say that groups $\mathfrak{G}_1$ and $\mathfrak{G}_2$ from $S(X)$ have one and the same orbital type if there is a mapping f in $S(X)$ such that for every orbit X_α of the group $\mathfrak{G}_1$, $f(X_\alpha)$ is an orbit of $\mathfrak{G}_2$.

2. **Lemma.** Let a transitive group G from $S(X)$ be representable in the form $G = HF$, where H and F are elementwise permutable normal divisors of G, with H intransitive, and

$$X = \bigcup X_\alpha, \quad \alpha \in I, \tag{1}$$

is the decomposition of X into orbits of the group H. Construct the representations $r_\alpha : H \to S(X_\alpha)$, $r_\alpha(h) = h_\alpha = h|X_\alpha$, $h \in H$, where $h|X_\alpha$ is the restriction of h to X_α, $\alpha \in I$. Then the representations r_α are pairwise equivalent.

Proof. By virtue of a well-known theorem, (1) is a decomposition of X into systems of imprimitivity of the group G. Let α and β be arbitrary indices from I. Then there is a substitution f_β in F such that $f_\beta(X_\alpha) = X_\beta$. Put $\varphi = f_\beta|X_\alpha$.

For $x \in X_\alpha$, $h \in H$, one can write

$$r_\beta(h) f_\beta(x) = h_\beta f_\beta(x) = h f_\beta(x) = f_\beta h(x) = f_\beta h_\alpha(x) = f_\beta r_\alpha(h)(x).$$

Hence $r_\beta(h)\varphi = \varphi r_\alpha(h)$.

3. **Theorem.** Let X be an infinite set. Then a transitive subgroup G of the group $SF(X)$ cannot be represented in the form of a product of two proper elementwise permutable normal divisors. In particular, G is indecomposable into a direct product.

Proof. Let a transitive subgroup G of the group $SF(X)$ be representable in the form of the product $HF = G$ of its proper elementwise permutable normal divisors H and F. Then two cases are possible: 1) both groups H and F are transitive; 2) at least one of them is intransitive. We shall show that both cases lead to a contradiction. By virtue of Lemma 1 of the paper (^{1}), in the first case H and F are regular groups. The latter contradicts the inclusion $H \subset SF(X)$. Consider the second case. Let, for example, the group H be intransitive and (1) be the decomposition of the set X into orbits of the group H. Then, according to the lemma, the representations r_α, $\alpha \in I$, of the group-

the groups H are pairwise equivalent. Hence, and from the inclusion $H \subset SF(X)$, it follows that the set I is finite.

Consequently, the orbits of the group H are infinite, and the group H itself is infinite. The same arguments are applicable also to the group F; consequently, F is an infinite group. Since H is invariant in G, (1) is a decomposition of X into systems of imprimitivity of the group G.

Consider the homomorphism determined by this decomposition:

$$\gamma : G \to S_k, \tag{2}$$

where S_k is the symmetric group of degree k, and k is the cardinality of the set I. Let $N = \ker\gamma$, $\Phi = F \cap N$. Then $F : \Phi \leq k!$. Consequently, the group Φ is infinite. Put $H_\alpha = H|X_\alpha$, $\Phi_\alpha = \Phi|X_\alpha$; since the representations r_α are pairwise equivalent, the group H_α is infinite for every $\alpha \in I$. From the finiteness of I and the infinitude of the group Φ there follows the existence of such an $\alpha \in I$ that the group Φ_α is infinite. Choose α so that Φ_α is infinite.

Obviously, H_α is a transitive subgroup of $S(X_\alpha)$, and Φ_α is contained in the centralizer of the group H_α in $S(X_\alpha)$. If Φ_α is transitive, then we arrive at a contradiction as in the first case. Let Φ_α be intransitive and

$$X_\alpha = \bigcup Y_\beta, \quad \beta \in B \tag{3}$$

be the decomposition of X_α into orbits of the group Φ_α. According to Lemma 2 of article (^{1}), the representations $\rho_\beta : \Phi_\alpha \to S(Y_\beta)$, where $\rho_\beta(x) = x|Y_\beta$, are pairwise equivalent. Consequently, the group $F_{\alpha\beta} = \Phi_\alpha|Y_\beta$ is infinite, and the cardinality of B is finite.

Put $H_\alpha^\beta = \{h \mid h \in H_\alpha,\ h(Y_\beta) = Y_\beta\}$, $H_{\alpha\beta} = H_\alpha^\beta | Y_\beta$.

It is easy to see that (3) is a decomposition of X_α into systems of imprimitivity of the group H_α. Consequently, $H_{\alpha\beta}$ is a transitive subgroup of $S(Y_\beta)$; $F_{\alpha\beta}$ is also a transitive subgroup of $S(Y_\beta)$. The groups $H_{\alpha\beta}$ and $F_{\alpha\beta}$ commute elementwise. According to Lemma 1 of article (^{1}), $F_{\alpha\beta}$, $H_{\alpha\beta}$ are regular groups. But, on the other hand, $H_{\alpha\beta} \subset SF(Y_\beta)$, and in $SF(Y_\beta)$ there are no regular groups. The theorem is proved.

4. From the theorem just proved, in particular, there again follows the assertion of article (^{1}) that every transitive locally nilpotent subgroup of $SF(X)$ for infinite X is a p-group, and a maximal transitive locally nilpotent subgroup of $SF(X)$ is a Sylow p-subgroup of $SF(X)$.

I. D. Ivanyuta in (^{2}) proved that the transitive Sylow p-subgroups of $SF(X)$, for countable X, are pairwise conjugate in $SF(X)$, while for uncountable X there are no transitive Sylow p-subgroups in $SF(X)$. Hence, and from the preceding, it follows that the orbits of a locally nilpotent subgroup of the group $SF(X)$ are either countable or finite. In (^{3}) it is proved that, for finite X, the maximal transitive nilpotent subgroups of $S(X)$ are pairwise conjugate. Thus, up to conjugacy in $SF(X)$, a maximal locally nilpotent subgroup of $SF(X)$ is determined by its orbital type.

We note that the proposition of article (^{1}) concerning ZA-groups needs clarification. The correct formulation should read as follows. Every ZA-group contained in $SF(X)$ is a subdirect product of nilpotent groups.

The following problem suggests itself naturally. Let X be an infinite set, and let $\mathfrak{G}$ be an abstract group. Under what conditions does there exist an injective homomorphism $h : \mathfrak{G} \to SF(X)$ such that $\operatorname{Im} h$ is a transitive group?

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CITED LITERATURE

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[2] I. D. Ivanyuta, Dissertation, Institute of Mathematics, Academy of Sciences of the Ukrainian SSR, 1964.
[3] D. A. Suprunenko, DAN, **99**, 23 (1954).

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