THE SCHAUDER PRINCIPLE AND THE STABILITY OF PERIODIC SOLUTIONS

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Abstract

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MATHEMATICS

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THE SCHAUDER PRINCIPLE AND THE STABILITY OF PERIODIC SOLUTIONS

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This note studies the connection between the Schauder principle and the stability of periodic solutions of nonlinear differential equations. The investigation is based on the following two general theorems.

Let F be a completely continuous operator acting in a real Banach space E. A fixed point x_0 of this operator is called stable if, for every $\varepsilon > 0$, one can specify a $\delta > 0$ such that from the inequality $\|x - x_0\| \leq \delta$ there follow the inequalities $\|F^n x - x_0\| \leq \varepsilon$ $(n = 1, 2, ...)$. The point x_0 is called asymptotically stable if, in addition, $\|F^n x - x_0\| \to 0$ $(n \to \infty)$.

Assume first that $F\theta = \theta$ (θ is the zero of the space E) and that the operator F is representable in the form

$$Fx = Ax + Bx,$$

where A is a linear operator, while the operator B satisfies the condition

$$\|Bx_1 - Bx_2\| \leq q(r)\|x_1 - x_2\| \quad (\|x_1\|,\ \|x_2\| \leq r),$$

where $q(r) \to 0$ $(r \to 0)$. Further, suppose that 1 is a simple eigenvalue of the operator A, and that all other points of the spectrum of this operator lie inside the unit circle.

Theorem 1. *The point θ is asymptotically stable if and only if it is an isolated fixed point of the vector field*

$$\Phi x = x - Fx \tag{1}$$

and if its index is equal to 1.

The proof of the theorem is carried out by methods of the theory of cones ([1]). Along the way it is shown that the index of the fixed point θ of the vector field (1) can be equal only to 0, 1, or −1.

Assume now that the Banach space E is semi-ordered by means of some normal cone K. By $\langle x_1, x_2 \rangle$ we shall denote the set of elements x satisfying the inequalities

$$x_1 \leq x \leq x_2.$$

The set $\langle x_1, x_2 \rangle$ is called a conical segment.

Theorem 2. *Let an analytic operator F, monotone on the conical segment $\langle x_1, x_2 \rangle$, completely continuous, map $\langle x_1, x_2 \rangle$ into itself, i.e. $Fx_1 \geq x_1$ and $Fx_2 \leq x_2$, with $Fx_1 \neq x_1$ and $Fx_2 \neq x_2$. Let the derivative $F'(x)$ be a u_0-positive operator for every x representable in the form $x = Fy$, where $y \in \langle x_1, x_2 \rangle$.*

Then the operator F has at least one asymptotically stable fixed point $x_0 \in \langle x_1, x_2 \rangle$.

In the proof of the theorem, the Schauder principle and Theorem 1 are used.

We pass to applications.

1. Let Ω be a bounded open domain of the n-dimensional space E_n of points $x = (x_1, \dots, x_n)$, belonging to the class $A^{(2,\lambda)}$. In the domain Ω there is considered

consider the quasilinear parabolic equation

$$M_1 u \equiv \frac{\partial u}{\partial t} - \sum_{i,k=1}^{n} a_{ik}(t,x) \frac{\partial^2 u}{\partial x_i \partial x_k} + f\left(t, x, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) = 0, \qquad (2)$$

where

$$\sum_{i,k=1}^{n} a_{ik}(t,x) \xi_i \xi_k \geq \gamma_0 \sum_{i=1}^{n} \xi_i^2 \qquad (\gamma_0 > 0)$$

for all real ξ_i, $t, x \in [0, \omega] \times \Omega$. It is assumed that all the functions $a_{ik}(t,x)$, $\frac{\partial}{\partial x_j} a_{ik}(t,x)$, $f(t, x, u, v_1, \dots, v_n)$, and $\frac{\partial}{\partial x_j} f(t, x, u, v_1, \dots, v_n)$ satisfy, with respect to the variables $t, x \in [0, \omega] \times \overline{\Omega}$, a Hölder condition on each compact set with respect to $u, v_1, \dots, v_n$. Further, it is assumed that the function $f(t, x, u, v_1, \dots, v_n)$ is analytic in the variables $u, v_1, \dots, v_n$ and that the inequality

$$|f(t, x, u, v_1, \dots, v_n)| \leq C_1(R) + C_2(R) \sum_{i=1}^{n} |v_i|^2$$

holds, where $t, x \in [0, \omega] \times \overline{\Omega}$, $|u| \leq R$. Finally, it is assumed that the functions $a_{ik}(t, x)$ and $f(t, x, u, v_1, \dots, v_n)$ are ω-periodic in t.

Below, by a solution of equation (2) we shall mean any function $u(t, x)$ $(0 \leq t \leq T,\ x \in \overline{\Omega})$ that is continuous in the aggregate of variables $t, x \in [0, T] \times \overline{\Omega}$, that satisfies equation (2) for $t, x \in (0, T] \times \Omega$, and that, for $x \in \Gamma$, $0 \leq t \leq T$ (Γ is the boundary of the domain Ω), satisfies the zero boundary condition.

Let $u_0(t, x)$ $(-\infty < t < \infty,\ x \in \overline{\Omega})$ be some ω-periodic in t solution of equation (2). We shall call it Lyapunov stable if for every $\varepsilon > 0$ one can indicate a $\delta > 0$ such that:

a) for any initial condition $u(0, x)$ satisfying the inequality

$$\|u(0, x) - u_0(0, x)\|_{C^{(1,\nu)}(\Omega)} \leq \delta \qquad (0 < \nu < 1), \tag{3}$$

the Cauchy problem for equation (2) is globally solvable on $(0, \infty)$;

b) from inequality (3) there follows the inequality

$$\|u(t, x) - u_0(t, x)\|_{C^{(1,\nu)}(\Omega)} \leq \varepsilon \qquad (0 \leq t < \infty).$$

We shall call the solution $u_0(t, x)$ asymptotically Lyapunov stable if, in addition,

$$\lim \|u(t, x) - u_0(t, x)\|_{C^{(1,\nu)}(\Omega)} = 0 \qquad (t \to \infty).$$

Let us introduce into consideration two functions $\psi_1(t, x)$ and $\psi_2(t, x)$. Below it is assumed that these functions satisfy, in the aggregate of variables $t, x \in [0, \omega] \times \overline{\Omega}$, some Hölder condition, are continuously differentiable in t, and twice continuously differentiable in the spatial variables in the domain $(0, \omega] \times \Omega$.

Theorem 3. Suppose there exist functions $\psi_1(t, x)$ and $\psi_2(t, x)$ $(\psi_1(t, x) \leq \psi_2(t, x)$ for $t, x \in [0, \omega] \times \overline{\Omega})$ satisfying the inequalities

$$\begin{aligned}
&M_1\psi_1(t, x) \leq 0 \leq M_1\psi_2(t, x) && (0 < t \leq \omega,\ x \in \Omega),\\
&\psi_1(t, x) \leq 0 \leq \psi_2(t, x) && (0 \leq t \leq \omega,\ x \in \Gamma),\\
&\psi_1(0, x) \leq \psi_1(\omega, x), \quad \psi_2(0, x) \geq \psi_2(\omega, x) && (x \in \Omega).
\end{aligned}$$

Suppose that neither of the functions $\psi_1(0, x)$ and $\psi_2(0, x)$ is the initial condition of an ω-periodic solution of equation (2).

Then equation (2) has at least one Lyapunov-asymptotically stable ω-periodic in t solution $u_0(t, x)$, satisfying the inequalities

$$\psi_1(t, x) \leq u_0(t, x) \leq \psi_2(t, x) \qquad (0 \leq t \leq \omega,\ x \in \overline{\Omega}).$$

2. Consider the ordinary differential equation of second order

$$M_2 x \equiv \ddot{x} + f(t, x, \dot{x}) = 0. \tag{4}$$

We shall assume that the function $f(t, x, y)$, continuous in the aggregate of variables, is ω-periodic in t, analytic in the variables x and y, and satisfies the inequality

$$|f(t, x, y)| \leq C_1(R) + C_2(R)|y|^2 \qquad (0 \leq t \leq \omega,\ |x| \leq R).$$

Theorem 4. Let there exist twice continuously differentiable functions $x_1(t)$ and $x_2(t)$ ($x_1(t) \leq x_2(t)$ for $0 \leq t \leq \omega$), satisfying the inequalities

$$M_2 x_1(t) \geq 0 \geq M_2 x_2(t) \qquad (0 \leq t \leq \omega),$$

$$x_1(0) = x_1(\omega), \qquad \dot{x}_1(0) \leq \dot{x}_1(\omega), \qquad x_2(0) = x_2(\omega), \qquad \dot{x}_2(0) \geq \dot{x}_2(\omega).$$

Let neither of the vectors $\{x_1(0), \dot{x}_1(0)\}$ and $\{x_2(0), \dot{x}_2(0)\}$ be an initial condition of an ω-periodic solution of equation (4). Then equation (4) has at least one ω-periodic solution $x_0(t)$, unstable in both directions of time, satisfying the inequalities

$$x_1(t) \leq x_0(t) \leq x_2(t) \qquad (0 \leq t \leq \omega).$$

Theorem 4 complements the results of [2].

3. As a final example, consider the problem of the existence of stable periodic solutions for the equation of order m

$$M_3 x \equiv x^{(m)} + a_1(t) x^{(m-1)} + \ldots + a_m(t) x = f(t, x), \tag{5}$$

where the continuous functions $a_i(t)$ and $f(t, x)$ are ω-periodic in t. It is assumed below that the function $f(t, x)$ is analytic in the variable x and is nondecreasing in this variable. In addition, it is assumed that the multipliers of the equation $M_3 x = 0$ lie inside the unit circle and that the Cauchy function $K(t, \tau)$ of this equation is nonnegative for all $-\infty < \tau \leq t < \infty$.

Theorem 5. Let there exist m-times continuously differentiable ω-periodic functions $x_1(t)$ and $x_2(t)$ ($x_1(t) \leq x_2(t)$ for $0 \leq t \leq \omega$), such that the inequalities

$$M_3 x_1(t) \leq f[t, x_1(t)], \qquad M_3 x_2(t) \geq f[t, x_2(t)] \qquad (0 \leq t \leq \omega)$$

are satisfied. Let neither of the vectors $\{x_1(0), \ldots, x_1^{(m-1)}(0)\}$ and $\{x_2(0), \ldots, x_2^{(m-1)}(0)\}$ be an initial condition of an ω-periodic solution of equation (5).

Then equation (5) has at least one Lyapunov-asymptotically stable ω-periodic solution $x_0(t)$, satisfying the inequalities

$$x_1(t) \leq x_0(t) \leq x_2(t) \qquad (0 \leq t \leq \omega).$$

For $m = 2$, the restrictions on the linear part of equation (5) can be substantially weakened: it is sufficient to assume that the Cauchy function $K(t, \tau)$ is nonnegative for all $\tau \leq t \leq \tau + \omega$ and $0 \leq \tau \leq \omega$.

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2. H. W. Knobloch, *Math. Zs.*, **82**, 177 (1963).

Note: Figure translations are in progress. See original paper for figures.