ON THE QUESTION OF THE SCHWARZSCHILD METRIC IN A SYNCHRONOUS REFERENCE SYSTEM

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As is known, by the holonomic transformation

$$c\,dt = c\,d\tau \pm \frac{\sqrt{r_g/r}\,dr}{1 - r_g/r},$$

which gives

$$ct = c\tau + 2\sqrt{r_g r} \mp r_g \ln \frac{1 + \sqrt{r_g/r}}{1 - \sqrt{r_g/r}}, \tag{1}$$

where $r_g = 2Gm/c^2$ is the gravitational radius, the Schwarzschild interval (^{1,2})

$$-ds^2 = -c^2 dt^2 (1 - r_g/r) + \frac{dr^2}{1 - r_g/r} + r^2 d\Omega^2, \tag{2}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$, is brought to the form

$$-ds^2 = -c^2 d\tau^2 (1 - r_g/r) \mp 2\sqrt{r_g/r}\, c\, d\tau\, dr + dr^2 + r^2 d\Omega^2. \tag{3}$$

Further, by the holonomic transformation ($^{2-4}$)

$$r^{3/2} = f(R) \pm {}^3/_2 \sqrt{r_g}\, c\tau, \tag{4}$$

where $f(R)$ is for the time being a completely arbitrary function of R, the interval (3) is reduced to the form (a synchronous reference system)

$$-ds^2 = -c^2 d\tau^2 + {}^4/_9 (df/dR)^2 dR^2 + r^2 d\Omega^2. \tag{5}$$

Usually (3,5) it is assumed that

$$f(R) = {}^3/_2 R\sqrt{r_g}; \tag{6}$$

then $df/dR = f' = {}^3/_2\sqrt{r_g}$, and in this case (4) and (5) take the form

$$r^{3/2} = {}^3/_2\sqrt{r_g}(R \pm c\tau), \tag{7}$$

$$-ds^2 = -c^2 d\tau^2 + \frac{r_g}{r}\, dR^2 + r^2 d\Omega^2, \tag{8}$$

or, finally,

$$-ds^2 = -c^2 d\tau^2 + r_g^{2/3}\left[\frac{dR^2}{[{}^3/_2(R \pm c\tau)]^{2/3}} + \left[{}^3/_2(R \pm c\tau)\right]^{4/3} d\Omega^2\right]. \tag{9}$$

Transformations (1) and (4) are performed in order to eliminate the singularity at $r = r_g$ in the standard Schwarzschild metric (2).

In choosing $f(R)$ in the form (6), the aim is to obtain a "smooth" solution at $r = r_g$, which is indeed achieved, but in the limiting transition to a metric that does not take the gravitational field into account, i.e., at $G = 0$, $r_g = 0$, we obtain $ds = c\, d\tau$, i.e., the Galilean metric, whereas in the limiting transition in metric (2) we obtain the Minkowski metric

$$-ds^2 = -c^2 d\tau^2 + dr^2 + r^2 d\Omega^2, \tag{10}$$

with $t = \tau$.

Obviously, the apparent arbitrariness of the function $f(R)$ must be disposed of in such a way that, in the limiting transition (5) for $r_g = 0$, we would obtain the metric (10) (6). From (4) it follows that $f(R) = R^{3/2}$, $f' = {}^3/_2\sqrt{R}$, and (5) takes the form

$$-ds^2 = -c^2 d\tau^2 + \frac{R}{r}\, dR^2 + \left(\frac{r}{R}\right)^2 R^2 d\Omega^2, \tag{11}$$

where

$$r = \left[R^{3/2} \pm {}^3/_2\sqrt{r_g}\, c\tau\right]^{2/3}, \qquad r/R = \left(1 \pm {}^3/_2\sqrt{r_g/R}\, c\tau/R\right)^{2/3}. \tag{12}$$

It follows that (11) takes the form

$$-ds^2 = -c^2 d\tau^2 + \frac{dR^2}{\left(1 \pm {}^3\!/_2\sqrt{\frac{r_g}{R}\frac{c\tau}{R}}\right)^{2/3}} + \left(1 \pm {}^3\!/_2\sqrt{\frac{r_g}{R}\frac{c\tau}{R}}\right)^{4/3} R^2 d\Omega^2. \tag{13}$$

For $G = 0$, $r_g = 0$, $R = r$, and the metric (13) passes into the Minkowski metric (10). In this case the singularity at $r = r_g$ will likewise not occur, as when R is chosen in the form (6).

The metric

$$-ds^2 = -c^2 dt^2 (1 - r_g/r) + dr^2 + r^2 d\Omega^2, \tag{14}$$

approximately, by

$$c\,dt = c\,d\tau \pm \sqrt{\frac{r_g/r}{1 - r_g/r}}\,dr, \qquad d\left(R^{3/2} \pm {}^3\!/_2\sqrt{r_g}\,c\tau\right) = {}^3\!/_2\,dr\sqrt{r - r_g}, \tag{15}$$

$$r = r_g + \left[R^{3/2} \pm {}^3\!/_2\sqrt{r_g}\,c\tau\right]^{2/3}, \tag{16}$$

can be brought to the form (11). It follows that in a synchronous system Newton' s "space" is also "curved" and "tensorial," like Einstein' s space, a point to which, generally speaking, no attention is paid.

In the case of the metric giving the "linear" approximation of the Schwarzschild interval,

$$-ds^2 = -c^2 dt^2 (1 - r_g/r) + dr^2 (1 + r_g/r) + r^2 d\Omega^2, \tag{17}$$

by the transformations

$$c\,dt = c\,d\tau \pm \sqrt{\frac{r_g}{r}\frac{1 + r_g/r}{1 - r_g/r}}\,dr, \qquad d\left(R^{3/2} \pm {}^3\!/_2\sqrt{r_g}\,c\tau\right) = \frac{3}{2}\,dr\sqrt{r\left(1 - \frac{r_g^2}{r^2}\right)}, \tag{18}$$

approximately

$$r \simeq \left(R^{3/2} \pm \frac{3}{2}\sqrt{r_g}\,c\tau\right)^{2/3} - \frac{r_g^2}{\left[R^{3/2} \pm {}^3\!/_2\sqrt{r_g}\,c\tau\right]^{2/3}}, \tag{19}$$

the metric is likewise brought to the form (11).

It is interesting to note that, in the first approximation, the metric (11) is the same for all three cases and has the form

$$-ds^2 = -c^2 d\tau^2 + dR^2 \left(1 \mp \sqrt{r_g/R}\, c\tau/R\right) + \left(1 \pm 2\sqrt{r_g/R}\, c\tau/R\right) R^2 d\Omega^2. \quad (20)$$

For the "exact" interval (12), when $r^{3/2} = R^{3/2} \pm \frac{3}{2}\sqrt{r_g}\, c\tau$, on differentiating we obtain $\sqrt{r/R}\, dr/c\, d\tau = dR/c\, d\tau \pm \sqrt{r_g/R}$; for $r = \mathrm{const}$ we shall have $(dR/c\, d\tau)^2 = v^2/c^2 = r_g/R = 2GM/Rc^2$, whence it follows that $v^2 = 2GM/R$, and this in turn indicates that the "reference system" $(R;\tau)$ is as it were freely falling in the gravitational field of a particle of mass M.

It is of great interest to investigate an almost analogous interval with periodic motion, when we replace $c\tau/R$ by $\frac{R_0}{R}\cos\omega_0 t$, where $\omega_0 = c/R_0$, which in the first approximation for small τ, when $c\tau/R_0 =$

$= \omega_0\tau \ll 1$, again gives $c\tau/R$. In this case (if $r = \mathrm{const}$) $v^2 = 2GM\sin^2\omega_0\tau/R$.

Thus, for the interval (11) we shall have

$$r^{3/2} = R^{3/2} \pm {}^3/_2\sqrt{2r_{gR}0}\cos\omega_0\tau, \quad (21)$$

where, for convenience in subsequent calculations, we have introduced the coefficient 2 under the radical so that the mean value of the quantity $\langle 2r_g\cos^2\omega_0\tau\rangle = r_g$.

We shall now carry out transformations inverse to those by which we passed from the ordinary Schwarzschild interval to the interval (11). Thus, first, instead of R we introduce r; obviously, since $r = (R^{3/2} \pm {}^3/_2\sqrt{2r_{gR}0}\sin\omega_0\tau)^{2/3}$, then $\sqrt{R}\, dR = \sqrt{r}\, dr \mp 2\sqrt{r_g}\, c\, d\tau\sin\omega_0\tau$, and therefore (11) takes the form

$$-ds^2 = -c^2 d\tau^2\left(1 - 2\,\frac{r_g}{r}\sin^2\omega_0\tau\right) \mp$$

$$\mp 2\sqrt{\frac{2r_g}{r}\sin^2\omega_0\tau}\, c\, d\tau\, dr + dr^2 + r^2 d\Omega^2. \quad (22)$$

Since the mean value $\langle\sin^2\omega_0\tau\rangle = {}^1/_2$, (22) takes the form (3), whence, by the transformation $c\, d\tau = c\, dt \mp \dfrac{\sqrt{r_g/r}\, dr}{1 - r_g/r}$, we arrive at the ordinary Schwarzschild interval (2).

The indicated averaging procedure is somewhat primitive; it is more expedient to proceed as follows. Let $c\tau = ct + \psi(t, r)$; then the interval (22) takes the form:

$$-ds^2 = -c^2 dt^2 (1+\dot{\psi})^2 \left(1 - \frac{2r_g}{r}\sin^2\omega_0\tau\right) + r^2 d\Omega^2 \pm$$

$$\pm 2\left[\sqrt{2\,\frac{r_g}{r}\sin^2\omega_0\tau} + \psi'\left(1 - \frac{2r_g}{r}\sin^2\omega_0\tau\right)(1+\dot{\psi})\right] c\,dt\,dr +$$

$$+dr^2\left[1 \mp 2\psi'\sqrt{\frac{2r_g}{r}\sin^2\omega_0\tau} - \psi i'^2\left(1 - \frac{2r_g}{r}\sin^2\omega_0\tau\right)\right],$$

where $\dot{\psi} = \partial\psi/c\,\partial t$; $\psi' = \partial\psi/\partial r$.

We require that the mixed term $c\,dt\,dr$ vanish; then

$$\psi' = -\sqrt{\frac{2r_g}{r}\sin^2\omega_0\tau}\bigg/\left[1 - \frac{2r_g}{r}\sin^2\omega_0\tau\right], \tag{23}$$

where $\tau = t + \psi/c$, and the interval takes the form:

$$-ds^2 = -c^2 dt^2 (1+\dot{\psi})^2 \left(1 - 2\,\frac{r_g}{r}\sin^2\omega_0\tau\right) +$$

$$+dr^2\bigg/\left[1 + 2\,\frac{r_g}{r}\sin^2\omega_0\tau\right] + r^2 d\Omega^2. \tag{24}$$

Upon averaging, $\langle\sin^2\omega_0\tau\rangle = {}^1/_2$, $\psi = \psi(r)$, $\dot{\psi} = 0$, and the interval indeed takes the form (2).

We now write (23) in the form

$$\psi' = \frac{\partial\psi}{\partial r} = \frac{c\,\partial\tau}{\partial r} = -\sqrt{2\,\frac{r_g}{r}\sin^2\omega_0\tau}\bigg/\left[1 - \frac{2r_g}{r}\sin^2\omega_0\tau\right]. \tag{25}$$

This equation is plainly not integrable exactly, but approximate integration is easily performed. In the first approximation we have

$$c\,\frac{\partial\tau}{\partial r} = r_0\,\frac{\partial(\omega_0\tau)}{\partial r} = -\sqrt{2\,\frac{r_g}{r}}\,\sin\omega_0\tau;$$

whence it follows that

$$\ln \operatorname{tg} \frac{\omega_0 \tau}{2} = T(t) - \frac{2}{r_0}\sqrt{2r_g r} = \ln \operatorname{tg} \frac{\omega_0 t}{2} - \frac{2}{r_0}\sqrt{2r_g r} \tag{26}$$

($T(t)$ is determined from the condition $t = \tau$ for $r_g = 0$); differentiating with respect to ct, we obtain

$$\frac{\partial(\omega_0 \tau)}{\sin \omega_0 \tau \, c\, \partial t} = \dot{T} = \frac{1 + \dot{f}}{r_0 \sin \omega_0 \tau} = \frac{1}{r_0 \sin \omega_0 t}.$$

The metric (24) now takes the form

$$-ds^2 = -c^2 dt^2 r_0^2 \dot{T}^2 \sin^2 \omega_0 \tau \left[1 - 2\frac{r_g}{r} \sin^2 \omega_0 \tau\right] + \frac{dr^2}{1 - 2\frac{r_g}{r}\sin^2 \omega_0 \tau} + r^2 d\Omega^2. \tag{27}$$

In the same approximation, solution (26) may be put in the form

$$\operatorname{tg} \frac{\omega_0 \tau}{2} = \operatorname{tg} \frac{\omega_0 t}{2} \exp\left[-\frac{2}{r_0}\sqrt{2r_g r}\right] = \operatorname{tg} \frac{\omega_0 t}{2}\left(1 - \frac{2}{r_0}\sqrt{2r_g r}\right). \tag{28}$$

It follows from this that

$$\sin \omega_0 \tau = \sin \omega_0 t \left(1 - \frac{2}{r_0}\sqrt{2r_g r}\cos \omega_0 t\right).$$

In this case we shall have:

$$r_0^2 \dot{T}^2 \sin^2 \omega_0 \tau = \frac{\sin^2 \omega_0 \tau}{\sin^2 \omega_0 t} = \left(1 - \frac{2}{r_0}\sqrt{2r_0 r}\cos \omega_0 t\right)^2.$$

Averaging over time in the same approximation, we find

$$\left\langle r_0^2 \dot{T}^2 \sin^2 \omega_0 \tau \right\rangle = 1.$$

Now (27) will take the form

$$-ds^2 = -c^2 dt^2 \left(1 - 2\frac{r_g}{r}\sin^2 \omega_0 t\right) + \frac{dr^2}{1 - 2\frac{r_g}{r}\sin^2 \omega_0 t} + r^2 d\Omega^2. \tag{29}$$

Averaging over time, we obtain the Schwarzschild metric. Metric (29) may subsequently be convenient in quantizing the interval.

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Note: Figure translations are in progress. See original paper for figures.