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1969

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Abstract

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UDC 539.311

THEORY OF ELASTICITY

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A VARIATIONAL PRINCIPLE IN FRACTURE MECHANICS

(Presented by Academician L. I. Sedov on 26 VI 1968)

Usually the problem of macroscopic cracks developing in solids is posed for rectilinear cracks, with the line of crack propagation assumed to be given. This restriction can be removed if one considers a sequence of solutions of the problem of elasticity theory for identical bodies, each of which contains a certain cut (crack) of arbitrary configuration. This sequence constitutes a class of admissible functions, from which the particular solution corresponding to the equilibrium of a body with a crack is selected by means of the variational principle proposed here.

The basic energy equation for a solid in the case of development of internal ruptures under quasi-brittle fracture of the body has the form ⁽¹⁾

$$\delta E + \delta W + \delta U_0 = \delta A + \delta Q + \delta Q^*, \quad (1)$$

where E is the kinetic energy of the body; W is the internal elastic bulk energy, depending on the specific entropy and the components of the strain tensor; U_0 is the fracture energy; δA is the work of external forces; δQ is the heat influx; δQ^* is the external influx of energy due to special microscopic effects (chemical, etc.).

Let $\delta E = \delta Q = \delta Q^* = 0$, and let the body contain a linear crack of length l . Then

$$\delta U_0 - \delta A + \delta W = 0. \quad (2)$$

Let us rewrite relation (2) in two forms: one for a variation of the shape of the crack trajectory with its ends fixed, and the other for a variation of the position of the crack tip along a fixed trajectory. In the case when the crack trajectory is varied ⁽²⁾, we obtain the first relation

Fig. 1

Figure 1: Fig. 1

$$\delta \int_{l_1+l_2} \gamma dS + (-\delta A + \delta W) = 0. \quad (3)$$

The second condition—for the crack tip—is

$$2\gamma + \frac{\partial}{\partial l}(-A + W) \leq 0, \quad (4)$$

where γ is the surface density of fracture energy (taken here as constant); l_1, l_2 are the opposite surfaces of the crack.

The equality sign in (4) refers to equilibrium cracks and to the instant of initiation (in some cases, termination) of the motion of nonequilibrium ones; the inequality sign corresponds to the case when the crack is nonequilibrium ⁽³⁾.

If one makes use of the known representation of a body with a crack and an external load as the sum of two terms ⁽⁴⁾, then it can be shown ⁽⁵⁾ that the variation of the potential energy of the system in (3), with fixed—

at the crack ends will be

$$-\delta A + \delta W = -\frac{1}{2} \delta \int_{l_1+l_2} \mathbf{p} \mathbf{u} dS, \quad (5)$$

where \mathbf{p} is the stress vector on elements whose position coincides with the surface of the crack, but these stresses are calculated for the loaded body that does not contain the crack, and are taken with the opposite sign; the components of the vector are determined in the usual way ($p_j = \sigma_{ij} n_i$) from the stresses in the body without a crack under the action of the prescribed loads and the direction cosines of the normal to the surface of the crack being sought; \mathbf{u} is the displacement vector of the points of the crack surface in the body on which no prescribed external loads act, but which is, however, loaded on the crack surface by the indicated stresses.

Fig. 1

Briefly, the essence of the derivation of formula (5) reduces to the following. Let us imagine a body with a cut, which is in two equilibrium states 1 and 2 (Fig. 1) under the same external load. These two states differ from one another in that in state 2 the trajectory of the cut has been varied (in state 1 the trajectory of the cut is described by the equation $\eta = \eta(\xi)$, and in state 2 by $\eta_* = \eta(\xi) + \delta\eta(\xi)$). It is required to determine the quantity $-\delta A + \delta W$ arising in connection with the indicated variation of the trajectory.

We replace each of the two states 1 and 2 by two: one state with the prescribed loads but without a cut, and another without loads but with a cut, on the surface of which there act stresses taken with the opposite sign from the continuous body (Fig. 1). Such a replacement is possible on the basis of the superposition principle of the linear theory of elasticity. All these states satisfy the conditions of mechanical equilibrium.

Let us find the value of $-\delta A + \delta W$ for states 5, 6 and 3, 4 instead of 1 and 2. States 3 and 5 are identical; therefore the corresponding variation $-\delta A + \delta W$ is absent. The forces acting on the surface of the cut in states 6 and 4 are represented by the stresses of the continuous body, which are determined by the values of the external loads. Therefore the desired quantity $-\Delta A + \Delta W$ can be computed as a difference, i.e.

$$-\Delta A + \Delta W = \left[- \int_{\Omega} (\mathbf{p} + \delta \mathbf{p})(\mathbf{u} + \delta \mathbf{u}) d\Omega + \frac{1}{2} \int_{\Omega} (\mathbf{p} + \delta \mathbf{p})(\mathbf{u} + \delta \mathbf{u}) d\Omega \right]_6 - \left[- \int_{l_1+l_2} \mathbf{p} \mathbf{u} dS + \frac{1}{2} \int_{l_1+l_2} \mathbf{p} \mathbf{u} dS \right]_4 \quad (\Omega = l_1 + l_2 + \delta l_1 + \delta l_2).$$

Hence we obtain formula (5).

In an analogous way to (5), the value of the released elastic energy is found in the second case—when the position of one of the crack tips is varied along a fixed trajectory (here, for convenience, the integrals are taken along one of the crack surfaces, under the assumption that the displacements of the opposite faces are equal)

$$-\delta A + \delta W = -\delta l \int_0^{l+\delta l} \mathbf{p} \frac{\partial \mathbf{u}}{\partial l} dS - \int_l^{l+\delta l} \mathbf{p} \mathbf{u} dS. \quad (6)$$

whence it follows that *

$$\frac{\partial}{\partial l} (-A + W) = - \int_0^l \mathbf{p} \frac{\partial \mathbf{u}}{\partial l} dS. \quad (7)$$

Let us note that the right-hand side of formula (6) can be given another representation, namely, in terms of the external macroscopic energy flux $\delta A_{\delta l}$ arising as a result of crack propagation into a region with an infinite stress concentration and expressed (in contrast to the right-hand side of formula (6)) through the characteristics of the states at the crack tip **. This flux was first obtained by Irwin (7) and, in (1), in a somewhat more general form.

Substituting (5) into (3) and (7) into (4), we obtain the following relations (the second is written for equilibrium cracks):

$$\delta \int_{l_1+l_2} (2\gamma - \mathbf{p}\mathbf{u}) dS = 0, \quad (8)$$

$$2\gamma - \int_0^l \mathbf{p} \frac{\partial \mathbf{u}}{\partial l} dS = 0. \quad (9)$$

It is not difficult to see that these equations are insufficient for determining the trajectory, since it is necessary to know the displacements of the points of the crack surface. To determine the displacements, one should add to these relations the Lagrange variational equation for a body free from prescribed loads but with a crack on whose surface stresses $-\mathbf{p}$ act,

$$\int_{l_1+l_2} \mathbf{p}\delta\mathbf{u} dS - \delta W_* = 0, \quad (10)$$

where W_* is the potential energy of the body with the load distributed over the crack surface.

The simultaneous solution of (8) and (10) is intended to determine both the crack trajectory and the displacement field $\mathbf{u}\{x, y, \Psi[\eta(\xi), \xi_A, \xi_B, \eta_A, \eta_B]\}$ in the cracked body ***. The functional Ψ reflects the influence of the crack shape on the displacement field. From the family of extremal curves one can select a particular solution, for example, by the point of crack initiation and by the value of the derivative at this point, obtained on the basis of the classical theories of strength.

* The second term of the right-hand side of (6) is of order $(\delta l)^{1/2}$ and is subsequently discarded.

** In our case it is possible to write $\delta A_{\delta l}$ through nonlocal characteristics:

$$\delta A_{\delta l} = -\delta l \int_0^l \mathbf{p} \frac{\partial \mathbf{u}}{\partial l} dS.$$

*** Coordinates of points off the trajectory are x, y , and on the trajectory ξ, η ; the coordinates of the crack ends have indices A and B ; the equation of the crack trajectory is $\eta = \eta(\xi)$.

Condition (9) makes it possible to determine the position of the crack tip on a known trajectory as a function of the magnitude of the loading parameter. An increase in the loading parameter (under proportional loading) causes the crack

tip to move along its trajectory, i.e., the crack propagates over the surface of the body.

The use of equation (9) in various crack problems [7] has shown agreement of the results with those that follow from the Griffith-Irwin theory.

The author expresses his gratitude to L. S. Polyak, L. I. Sedov, and A. S. Shvarts, and also gratefully acknowledges the assistance of Ya. B. Fridman, which was expressed in useful discussion of the work.

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Received
20 VI 1968

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