

**SOLUTION OF THE
CAUCHY PROBLEM
AND ITS
ASYMPTOTICS AS
 $(t \rightarrow \infty)$ FOR THE
LINEARIZED
EQUATIONS OF A
ROTATING VISCOUS
FLUID**

MATHEMATICAL PHYSICS

1969

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.83138>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.947

MATHEMATICAL PHYSICS

V. N. MASLENNIKOVA

SOLUTION OF THE CAUCHY PROBLEM AND ITS ASYMPTOTICS AS $t \rightarrow \infty$ FOR THE LINEARIZED EQUATIONS OF A ROTATING VISCIOUS FLUID

(Presented by Academician S. L. Sobolev on 30 V 1969)

We consider the linearized Navier-Stokes system for an incompressible rotating viscous fluid:

$$\frac{\partial \mathbf{v}}{\partial t} - [\mathbf{v}, \omega] + \nu \Delta \mathbf{v} + \text{grad } p = \mathbf{f}, \quad \text{div } \mathbf{v} = 0, \quad (1)$$

where $\mathbf{v}(x, t) = (v_1, v_2, v_3)$, $x = (x_1, x_2, x_3)$, $\omega = (0, 0, \omega)$, $\omega = \text{const} > 0$, $[\cdot, \cdot]$ is the vector product, $\nu = \text{const} > 0$.

In the paper the fundamental solution of the Cauchy problem is constructed for system (1) with initial data

$$\mathbf{v}(x, t)|_{t=0} = \mathbf{v}^0(x); \quad \text{div } \mathbf{v}^0(x) = 0 \quad (2)$$

and the influence of the term $[\mathbf{v}, \omega]$ on the asymptotic behavior of the solution as $t \rightarrow \infty$ is investigated.

We shall consider the homogeneous system (1) ($\mathbf{f} \equiv 0$). The solution of the nonhomogeneous system can be obtained by the known method (2). Let us note that, without taking rotation into account, the fundamental solutions of system (1) were constructed by Oseen (4).

For $\nu = 0$, i.e. for the Sobolev system (1), it was proved in (2) that the solution of the Cauchy problem decreases as $O(1/t)$. In (3) this question was also studied for the Sobolev system with compressibility taken into account. It turns out that, for system (1) with viscosity included, under certain natural conditions on the initial data, the vector $\mathbf{v}(x, t)$ —the solution of the homogeneous system (1)—has, as $t \rightarrow \infty$, the asymptotics $O(1/t^{5/2})$, i.e. the degree of decay of the solution is composed of the degree of decay of the solution for system (1) with

viscosity, without taking rotation terms into account, and the degree of decay of the solution caused by the term $[\mathbf{v}, \omega]$ in the Sobolev system.

Thus, let us find an explicit representation of the solution of the homogeneous system (1) with initial conditions (2). Assuming that the initial data and the solution admit a Fourier transform, we obtain the solution of our problem (in the Fourier transform) in the form

$$\tilde{\mathbf{v}}(\alpha, t) = \tilde{\mathbf{v}}^0(\alpha) \exp(-\nu|\alpha|^2 t) \cos \frac{\omega\alpha_3 t}{|\alpha|} + [\tilde{\mathbf{v}}^0, \alpha] \exp(-\nu|\alpha|^2 t) \frac{1}{|\alpha|} \sin \frac{\omega\alpha_3 t}{|\alpha|},$$

$$\tilde{p}(\alpha, t) = [\tilde{\mathbf{v}}^0, \alpha]_3 \frac{i\omega}{|\alpha|^2} \exp(-\nu|\alpha|^2 t) \cos \frac{\omega\alpha_3 t}{|\alpha|} - \tilde{v}_3^0(\alpha, t) \frac{i\omega}{|\alpha|} \exp(-\nu|\alpha|^2 t) \sin \frac{\omega\alpha_3 t}{|\alpha|},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha|^2 = \sum_{i=1}^3 \alpha_i^2$, and $[\tilde{\mathbf{v}}, \alpha]_3$ is the third component of the vector product.

In the (x, t) -representation the solution of the problem will have the form

$$\mathbf{v}(x, t) = \int \{ \mathbf{v}^0(y) K_1(x - y, t) + [\mathbf{v}^0(y), \mathbf{K}_2(x - y, t)] \} dy, \quad (3)$$

$$p(x, t) = \int \{ v_1^0(y) K_{32}(x - y, t) - v_2^0(y) K_{31}(x - y, t) - v_3^0(y) K_4(x - y, t) \} dy, \quad (4)$$

where

$$\begin{aligned} K_1(x, t) &= \frac{1}{(2\pi)^3} \int \exp[i(\alpha, x)] \exp(-\nu|\alpha|^2 t) \cos \frac{\omega\alpha_3 t}{|\alpha|} d\alpha, \\ K_2(x, t) &= \frac{1}{(2\pi)^3} \int \exp[i(\alpha, x)] \frac{\alpha}{|\alpha|} \exp(-\nu|\alpha|^2 t) \sin \frac{\omega\alpha_3 t}{|\alpha|} d\alpha, \quad (5) \\ K_{3j}(x, t) &= \frac{1}{(2\pi)^3} \int \exp[i(\alpha, x)] \frac{i\omega\alpha_j}{|\alpha|^2} \exp(-\nu|\alpha|^2 t) \cos \frac{\omega\alpha_3 t}{|\alpha|} d\alpha, \\ K_4(x, t) &= \frac{1}{(2\pi)^3} \int \exp[i(\alpha, x)] \frac{i\omega}{|\alpha|} \exp(-\nu|\alpha|^2 t) \sin \frac{\omega\alpha_3 t}{|\alpha|} d\alpha. \end{aligned}$$

Using Bochner's formula for multiple Fourier integrals, then passing to polar coordinates and using Sonine's formulas⁽⁵⁾ for cylindrical functions, the kernels (5) can be represented in the form

$$K_1(x, t) = \frac{1}{2(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \lambda^2 \exp(-\nu t \lambda^2) \frac{J_{1/2}(\Omega)}{\Omega^{1/2}} d\lambda,$$

$$\mathbf{K}_2(x, t) = \mathbf{K}'_2(x, t) + \mathbf{K}''_2(x, t),$$

$$\mathbf{K}'_2(x, t) = \frac{\mathbf{r}}{2(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \lambda^3 \exp(-\nu t \lambda^2) \frac{J_{3/2}(\Omega)}{\Omega^{3/2}} d\lambda,$$

$$\mathbf{K}''_2(x, t) = \left(0, 0, \frac{\omega t}{2(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \lambda^2 \exp(-\nu t \lambda^2) \frac{J_{3/2}(\Omega)}{\Omega^{3/2}} d\lambda \right), \quad (6)$$

$$K_{3j}(x, t) = -\frac{\omega x_j}{2(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \lambda^2 \exp(-\nu t \lambda^2) \frac{J_{3/2}(\Omega)}{\Omega^{3/2}} d\lambda,$$

$$K_4(x, t) = \frac{\omega}{2(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \lambda \exp(-\nu t \lambda^2) \frac{J_{1/2}(\Omega)}{\Omega^{1/2}} d\lambda,$$

where

$$\Omega = \sqrt{\lambda^2 r^2 + 2\lambda x_3 \omega t + \omega^2 t^2}, \quad r^2 = \sum_{i=1}^3 x_i^2.$$

The purpose of further transformations of the kernels K is to represent them in such a form that the nature of the interaction of the viscosity of the fluid and its oscillations, which are due to the term $[\mathbf{v}, \omega]$, is visible. We note that in system (1), when $\omega = 0$, the kernel

$$K_1(x, t) = \frac{1}{8(\pi \nu t)^{3/2}} \exp\left(-\frac{r^2}{4\nu t}\right),$$

while the remaining kernels are equal to zero. In Sobolev' s system ^(1,2) (when $\nu = 0$), the fundamental solutions are written in terms of Bessel functions with argument $\rho \omega t / r$, and the principal fundamental solution ⁽²⁾, with a weak singularity, has the form

$$\frac{1}{r} J_0\left(\rho \frac{\omega t}{r}\right), \quad \text{where } \rho^2 = x_1^2 + x_2^2, \quad r^2 = \rho^2 + x_3^2.$$

With the aid of the formula for Bessel functions ⁽⁵⁾

$$\frac{J_\mu \left(a \sqrt{\lambda^2 - 2\lambda\tau \cos \psi + \tau^2} \right)}{\left(\sqrt{\lambda^2 - 2\lambda\tau \cos \psi + \tau^2} \right)^\mu} = \frac{a^\mu}{\pi \cdot 2^\mu \Gamma(\mu)} \int_0^\pi d\varphi \int_0^\pi \exp\{ai[\lambda \cos \varphi - \tau(\cos \varphi \cos \psi - \sin \varphi \sin \psi \cos \chi)]\} \sin^{2\mu-1} \varphi$$

In (6) we carry out the integration with respect to λ and obtain the representation of the kernels (6) in the form:

$$K_1(x, t) = \frac{1}{8(\pi\nu t)^{3/2}} \int_0^{\pi/2} \left(1 - 2 \frac{r^2 \cos^2 \varphi}{4\nu t} \right) \exp \left(- \frac{r^2 \cos^2 \varphi}{4\nu t} - i \frac{\omega t x_3}{r} \cos \varphi \right) J_0 \left(\frac{\omega t \rho}{r} \sin \varphi \right) d\varphi,$$

$$K_2'(x, t) = \frac{rxi}{(2\pi)^{3/2} (2\nu t)^{5/2}} \int_0^{\pi/2} \cos \varphi \exp \left(- \frac{r^2 \cos^2 \varphi}{4\nu t} - i \frac{\omega t x_3}{r} \cos \varphi \right) \left(3 - \frac{r^2 \cos^2 \varphi}{2\nu t} \right) \frac{J_1 \left(\frac{\omega t \rho}{r} \sin \varphi \right)}{\frac{\omega t \rho}{r} \sin \varphi} d\varphi,$$

$$K_{23}''(x, t) = \frac{\omega t}{(2\pi)^{3/2} (2\nu t)^{3/2}} \int_0^{\pi/2} \left(1 - 2 \frac{r^2 \cos^2 \varphi}{4\nu t} \right) \exp \left(- \frac{r^2 \cos^2 \varphi}{4\nu t} - i \frac{\omega t x_3}{r} \cos \varphi \right) \frac{J_1 \left(\frac{\omega t \rho}{r} \sin \varphi \right)}{\frac{\omega t \rho}{r} \sin \varphi} d\varphi,$$

$$K_{3j}(x, t) = - \frac{\omega x_j}{(2\pi\nu t)^{3/2}} \int_0^{\pi/2} \left(1 - 2 \frac{r^2 \cos^2 \varphi}{4\nu t} \right) \exp \left(- \frac{r^2 \cos^2 \varphi}{4\nu t} - i \frac{\omega t x_3}{r} \cos \varphi \right) \frac{J_1 \left(\frac{\rho \omega t}{r} \sin \varphi \right)}{\frac{\rho \omega t}{r} \sin \varphi} d\varphi,$$

$$K_4(x, t) = - \frac{\omega ir}{8(\pi\nu t)^{3/2}} \int_0^{\pi/2} \cos \varphi \exp \left(- \frac{r^2 \cos^2 \varphi}{4\nu t} - i \frac{\omega t x_3}{r} \cos \varphi \right) J_0 \left(\frac{\omega t \rho}{r} \sin \varphi \right) d\varphi.$$

It can be proved (in (7) this is done for the first boundary-value problem) that the solution of the Cauchy problem for system (1) is unique in the class

$$W_{2,x,t}^{1,0}(E_3 \times (0 \leq t < \infty)).$$

The solution constructed, under weak assumptions concerning the vector $\mathbf{v}^0(x)$ (for example, $\mathbf{v}^0(x) \in L_2$), will belong to this uniqueness class.

The following theorem describes the asymptotic character of the behavior of our solution as $t \rightarrow \infty$.

Theorem. If the initial data $\mathbf{v}^0(x) \in L_1(E_3)$ and satisfy the compatibility condition $\text{div } \mathbf{v}^0 = 0$, then the solution of the homogeneous system (1) as $t \rightarrow \infty$ has the following asymptotics: uniformly for all $x \in E_3$, the vector $\mathbf{v}(x, t)$ decreases as $O(1/t^{5/2})$, and the function $p(x, t)$ as $O(1/t^2)$.

Proof. Replacing in the kernel $K_1(x - y, t)$ the Bessel function of half-integer order by the trigonometric one and making the substitution $\lambda_1 = \sqrt{\nu t} \lambda$, we have

$$K_1(x, t) = \frac{1}{4\pi^2(\nu t)^{3/2}\omega t} \int_{-\infty}^{+\infty} \lambda_1^2 e^{-\lambda_1^2} \frac{\sin\left(\omega t \sqrt{r^2 \lambda_1^2 / (\omega t)^2 (\nu t) + 2x_3 \lambda_1 / \omega t \sqrt{\nu t} + 1}\right)}{\sqrt{r^2 \lambda_1^2 / (\nu t) (\omega t)^2 + 2x_3 \lambda_1 / \omega t \sqrt{\nu t} + 1}} d\lambda_1. \quad (7)$$

In view of the absolute convergence of the integral in (7), we obtain the following estimate of the kernel, uniform for all $x \in E_3$, $t > 0$:

$$|K_1(x, t)| \leq c_1 / (\nu t)^{3/2} \omega t.$$

We note that at the origin of coordinates ($x = 0$) the kernel K_1 is computed exactly:

$$K_1(0, t) = \sin \omega t / 8\pi^{3/2} (\nu t)^{3/2} (\omega t).$$

For the remaining kernels, by analogous arguments we obtain uniform estimates of the form:

$$|K_2'(x, t)| \leq \frac{c_2'}{(\nu t)^2 (\omega t)^2}; \quad |K_{23}''(x, t)| \leq \frac{c_{23}''}{(\nu t)^{3/2} \omega t},$$

$$|K_{3j}(x, t)| \leq \frac{c_3}{(\nu t)^{3/2} (\omega t)^2}; \quad |K_4(x, t)| \leq \frac{c}{\nu t \omega t}.$$

Thus we have:

$$|\mathbf{v}(x, t)| \leq \frac{c}{(\nu t)^{3/2} \omega t} \|\mathbf{v}^0\|_{L_1(E_3)} + O\left(\frac{1}{t^4}\right) \|\mathbf{v}^0\|_{L_1(E_3)},$$

$$|p(x, t)| \leq \frac{c}{\nu t \omega t} \|v_3^0\|_{L_1(E_3)} + O\left(\frac{1}{t^{1/2}}\right) (\|v_1^0\|_{L_1(E_3)} + \|v_2^0\|_{L_1(E_3)}).$$

The theorem is proved.

Remark 1. If $\omega = 0$, $\nu > 0$, then the vector $\mathbf{v}(x, t)$ decreases as $O(1/(\nu t)^{3/2})$, while the function $p(x, t)$ is identically equal to zero (because of the solenoidality of the initial data).

Remark 2. If $\nu = 0$, $\omega \neq 0$, then, in view of the divergence of the integrals (6), the solution of the Cauchy problem must first be found for smooth initial data,

and then the smoothness of the initial data must be dispensed with by passing to singular integrals, as was done in ⁽²⁾. It is also shown there that the solution of the Cauchy problem will then decrease as $O(1/\omega t)$.

Remark 3. V. P. Maslov drew my attention to the fact that in the case of isotropic turbulence the decay law of M. D. Millionshchikov ⁽⁶⁾, when the viscosity of the fluid is substantial, has order $O(1/t^{5/2})$. The vector $\mathbf{v}(x, t)$ decays at this rate in our case as well. Apparently, there is an internal connection here.

Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
29 V 1969

REFERENCES CITED

- ¹ S. L. Sobolev, *Izv. AN SSSR, Ser. Mat.*, **18**, No. 1 (1954).
- ² V. N. Maslennikova, *Tr. Mat. Inst. im. V. A. Steklova AN SSSR*, **103** (1968).
- ³ V. N. Maslennikova, *DAN*, **187**, No. 5 (1969).
- ⁴ C. W. Oseen, *Hydrodynamik*, Leipzig, 1927.
- ⁵ N. Ya. Sonin, *Investigations on Cylindrical Functions and Special Polynomials*, Moscow, 1954.
- ⁶ L. D. Landau, E. M. Lifshitz, *Mechanics of Continuous Media*, Moscow, 1953.
- ⁷ V. N. Maslennikova, *Sibirsk. Mat. Zh.*, **2**, No. 5 (1961).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.