SPATIAL DOUBLY PERIODIC PROBLEMS OF THE THEORY OF ELASTICITY

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Abstract

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SPATIAL DOUBLY PERIODIC PROBLEMS OF THE THEORY OF ELASTICITY

Plane doubly periodic problems of the theory of elasticity were considered in works ($^{1-2,6}$).

In the present article a method is proposed for solving boundary-value problems of the theory of elasticity for a space perforated by cylindrical cavities whose axes are parallel, and a complete investigation is given of the resulting infinite systems of algebraic equations. The cross-section of the elastic body under consideration is a plane with circular holes of radius R, whose centers are located at the nodes of an oblique lattice. Place at the center of each of the holes the origin O_{qs} of a cylindrical coordinate system $(r_{qs}, \theta_{qs}, z_{qs})$, with the axes z_{qs} coinciding with the longitudinal axes of the cavities. Introduce the following notation: Γ_{qs} is the surface of the qs-th cavity $(q, s = 0, \pm1, \pm2, ...)$; R_{qs}, Θ_{qs} are the polar coordinates of the pole O_{00} in the qs-th coordinate system; $Z_{qs} = R_{qs} \cdot \exp(i\Theta_{qs})$. In what follows we shall regard the system with pole O_{00} as the basic one, omitting the lower indices in its coordinates.

Suppose that the components of displacement or stress prescribed on Γ_{qs} are periodic functions of the variable z.

To solve the formulated problem we use the general solution of Lamé’s equations in the form proposed by B. G. Galerkin (^{3}):

$$u_r = -\frac{1+\nu}{E}\left[\frac{\partial}{\partial r}\Phi - 2(1-\nu)\left(\cos\theta\,\nabla^2\psi_1 + \sin\theta\,\nabla^2\psi_2\right)\right],$$

$$u_\theta = -\frac{1+\nu}{E}\left[\frac{1}{r}\frac{\partial}{\partial \theta}\Phi + 2(1-\nu)\left(\sin\theta\,\nabla^2\psi_1 - \cos\theta\,\nabla^2\psi_2\right)\right], \qquad (1)$$

$$u_z = -\frac{1+\nu}{E}\left[\frac{\partial}{\partial z}\Phi - 2(1-\nu)\nabla^2\psi_3\right],$$

where ψ_i $(i = 1, 2, 3)$ are biharmonic functions, and

$$\Phi = \cos\theta\,\frac{\partial\psi_1}{\partial r} - \frac{1}{r}\sin\theta\,\frac{\partial\psi_1}{\partial\theta} + \sin\theta\,\frac{\partial\psi_2}{\partial r} + \frac{1}{r}\cos\theta\,\frac{\partial\psi_2}{\partial\theta} + \frac{\partial\psi_3}{\partial z}. \tag{2}$$

The functions ψ_i must satisfy the conditions of double periodicity, which is achieved by the representation

$$\psi_1 = \sin\chi z \sum_{q,s}\sum_{n=0}^{\infty} r_{qs} K_n(\chi r_{qs}) \left[A_n \cos(n+1)\theta_{qs} + B_n \sin(n+1)\theta_{qs}\right],$$

$$\psi_2 = \sin\chi z \sum_{q,s}\sum_{n=0}^{\infty} r_{qs} K_n(\chi r_{qs}) \left[-B_n \cos(n+1)\theta_{qs} + A_n \sin(n+1)\theta_{qs}\right],$$

$$\psi_3 = \cos\chi z \sum_{q,s}\sum_{n=0}^{\infty} \left[K_n(\chi r_{qs}) \left(C_n \cos n\theta_{qs} + D_n \sin n\theta_{qs}\right) + \right. \tag{3}$$

$$\left. + r_{qs} K_{n+1}(\chi r_{qs}) \left(E_n \cos n\theta_{qs} + F_n \sin n\theta_{qs}\right)\right],$$

where χ is a certain constant number, and the indices q and s vary from $-\infty$ to $+\infty$.

The arbitrary constants $A_n, B_n, \ldots, F_n$ can be determined from the boundary conditions on any of the surfaces Γ_{qs}, for example on Γ_{00}. For this it is necessary to have expressions for ψ_i in the coordinates (r, θ, z), which can be obtained by using consequences of Macdonald' s addition theorem:

$$K_n(\chi r_{qs}) e^{in\theta_{qs}} = \sum_p (-1)^p K_{n-p}(\chi R_{qs}) e^{i(n-p)\Theta_{qs}} I_p(\chi r) e^{ip\theta},$$

$$r_{qs} K_n(\chi r_{qs}) e^{i(n+1)\theta_{qs}} = \sum_p (-1)^p e^{i(n-p)\Theta_{qs}} \left[Z_{qs} K_{n-p}(\chi R_{qs}) I_p(\chi r) - e^{i\Theta_{qs}} K_{n-p+1}(\chi R_{qs}) r I_{p-1}(\chi r)\right] e^{ip\theta}, \tag{4}$$

$$r_{qs} K_{n+1}(\chi r_{qs}) e^{in\theta_{qs}} = \sum_p (-1)^p e^{i(n-p)\Theta_{qs}} \left[\overline{Z}_{qs} K_{n-p+1}(\chi R_{qs}) e^{i\Theta_{qs}} I_p(\chi r) - K_{n-p}(\chi R_{qs}) r I_{p+1}(\chi r)\right] e^{ip\theta} \quad (r < R$$

Thus, the transformation of the functions ψ_i to the coordinates (r, θ, z) is carried out with the aid of the transition formulas (4), after which the displacements can be written by formulas (1), and hence also the stresses in these coordinates.

Satisfying the boundary conditions on one of the cavities, we arrive at an infinite system of algebraic equations, which can be represented in the form

$$A_n^{(*)} X_n + \sum_{p=0}^{\infty} a_{n,p} X_p = b_n \quad (n = 0, 1, \ldots), \tag{5}$$

where X_n is a column vector with elements $A_n, B_n, \ldots, F_n$; $A_n^{(*)}$ and $a_{n,p}$ are matrices depending on the boundary conditions; b_n is a column vector of the Fourier coefficients of the prescribed boundary values of the displacement or stress components.

To reduce system (5) to canonical form, we make in it the change of unknowns, putting $A_n^{(*)} X_n = Y_n$:

$$Y_n + \sum_{p=0}^{\infty} a'_{n,p} Y_p = b_n \quad (n = 0, 1, \ldots). \tag{6}$$

For large values of the indices, the elements of the matrix of system (6) have the order of magnitude

$$\frac{(n+1)^{\gamma}(p+1)^{\gamma}(p+n)!}{n!\,p!}\left(\frac{R}{\delta}\right)^{n+p} = c_{n,p}, \tag{7}$$

where δ is the least distance between the axes of two neighboring cavities ($\delta > 2R$, since it is assumed that the surfaces Γ_{qs} do not touch), and γ is some constant number. This is proved with the aid of asymptotic formulas for modified cylindrical functions with large index [4], and also the inequality

$$\left|\sum_{q,s} K_{p-n}(\chi R_{qs}) e^{i(p-n)\Theta_{qs}}\right| < M\left[\left|R_{|p-n|,0}(i\chi\delta)\right| + \left|R_{|p-n|-1,1}(i\chi\delta)\right|\right],$$

where M is equal to the larger of the numbers

$$\left|\sum_{q,s} K_0(\chi R_{qs}) e^{i(p-n)\Theta_{qs}} \left(\frac{R_{qs}}{\delta}\right)^{-|p-n|+2l}\right| \quad \text{for } l \le \tfrac{1}{2}|p-n|,$$

$$\left|\sum_{q,s} K_1(\chi R_{qs}) e^{i(p-n)\Theta_{qs}} \left(\frac{R_{qs}}{\delta}\right)^{-|p-n|+2l+1}\right| \quad \text{for } l \le \tfrac{1}{2}(|p-n|-1),$$

and $R_{m,\nu}(z)$ are Lommel polynomials [4]. The order of the polynomials $R_{m,\nu}(z)$ for large values of the first index is determined from the limiting rela-

[[unclear: beginning of line]] Hurwitz (4)

$$\lim_{m\to\infty} \frac{\left(\frac{1}{2}z\right)^{\nu+m} R_{m,\nu+1}(z)}{\Gamma(\nu+m+1)} = J_\nu(z).$$

Since the double series $\sum\limits_{n,p=0}^{\infty} c_{n,p}$ converges, it follows from (7) that the series composed of the moduli of the coefficients $a'_{n,p}$ also converges, i.e., the determinant of the infinite system (6) belongs to the class of determinants of normal type (5). Thus, system (6) has a unique bounded solution if the corresponding homogeneous system has no solutions other than the zero solution (the free terms of system (6) are bounded). Assuming that the boundary-value problems formulated at the beginning of the paper have unique solutions, we arrive at the conclusion that the infinite system of algebraic equations (6) has a unique bounded solution. This solution can be obtained approximately from a truncated system, whose order depends on the distance between the axes of the cavities and on the required accuracy in satisfying the boundary conditions.

Remark. The proposed method can also be used to solve problems concerning a layer with a doubly periodic system of cylindrical cavities. If mixed boundary conditions of the type $u_\theta = 0$, $u_r = 0$, $\sigma_{zz} = 0$, or $u_z = 0$, $\tau_{rz} = 0$, $\tau_{\theta z} = 0$ are prescribed on the plane boundaries of the layer, then the problems are solved exactly. For this purpose, in expressions (3) it is necessary to choose the corresponding trigonometric functions of z and set $\kappa = \frac{\pi}{h} m$, where h is the thickness of the layer. If, however, the plane faces of the layer are free of stresses, then the functions ψ_i in the form (3) may be used for an approximate solution of the problems. In this case one condition, $\sigma_{zz} = 0$, is satisfied exactly, while the other two, $\tau_{rz} = 0$, $\tau_{\theta z} = 0$, are satisfied in the integral sense within one cell.

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Note: Figure translations are in progress. See original paper for figures.

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