THE LINEAR CONJUGATION PROBLEM FOR A HOLOMORPHIC VECTOR IN A BANACH SPACE

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Abstract

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MATHEMATICS

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THE LINEAR CONJUGATION PROBLEM FOR A HOLOMORPHIC VECTOR IN A BANACH SPACE

(Presented by Academician I. N. Vekua on 13 V 1969)

1. Let Ω^+, Ω^- be finite and infinite simply connected domains in the plane with common boundary C; for simplicity, assume that C satisfies the Lyapunov conditions.

Denote by $E_p(C)$ the Banach space whose elements are infinite sequences $x = \{x_n(t)\}$ of measurable functions on C satisfying the condition:

$$\|x\|_{E_p}^p = \sum_{n=1}^{\infty} \int_C |x_n(t)|^p \, |dt| < \infty, \qquad 1 < p < \infty. \tag{1}$$

$G(t) = \{g_{kl}(t)\}$, $k, l = 1, 2, ...$, is an infinite functional matrix whose elements are continuous functions, satisfying the following conditions: a) the matrix $G(t)$ represents a bounded operator in $E_p(C)$; b) the matrix $G(t)$ has a unique two-sided inverse, which is also a bounded operator in $E_p(C)$; c) the operator

$$\widetilde{K}\varphi \equiv \frac{1}{2\pi i} \int_C \frac{G(t) - G(t_0)}{t - t_0} \varphi(t) \, dt \tag{2}$$

is completely continuous in $E_p(C)$.

It is required to find a piecewise-holomorphic vector $\varphi^\pm(z) = \{\varphi_n^\pm(z)\}$, whose order at infinity does not exceed a certain number N, having almost everywhere on C angular boundary values $\varphi^\pm(t) \in E_p(C)$, satisfying the boundary condition

$$\varphi^+(t) = G(t)\varphi^-(t) + g(t), \tag{3}$$

where $g(t) \in E_p(C)$.

By the methods described in works ([1],[2]), the following result can be obtained:

Theorem 1. *Every solution of the homogeneous problem* (3) *(with $g \equiv 0$) is representable in the form*

$$\varphi(z) = c_1\omega^1(z) + \ldots + c_q\omega^q(z) + \sum_{k=1}^{\infty} \gamma_k \varphi^k(z), \tag{4}$$

where q is some finite number; $\omega^1(z), \ldots, \omega^q(z), \varphi^1, \varphi^2, \ldots$ are linearly independent particular solutions of problem (3), *with*

$$\lim z^{-N} \varphi_l^k(z) = \delta_{kl}, \qquad k, l = 1, 2, \ldots, \tag{5}$$

and the remaining ones (there will be finitely many of them) have order at infinity less than N; $c_1, \ldots, c_q, \gamma_1, \gamma_2, \ldots$ are arbitrary constants, where

$$\sum_{k=1}^{\infty} |\gamma_k|^2 < \infty.$$

In accordance with the generally accepted terminology, the problem

$$\psi^+(t) = [G^{-1}(t)]' \psi^-(t), \tag{6}$$

where $[G^{-1}(t)]'$ is the matrix inverse and transposed with respect to $G(t)$, will be called the adjoint of the homogeneous problem (3) (for $g(t) \equiv 0$). The index $\varkappa_G(E_p)$ is, as usual, the difference between the numbers of linearly independent solutions vanishing at infinity of the original homogeneous problem and of its adjoint.

Consequence. The index of problem (3) is finite.

We note that, in contrast to the finite-dimensional case, the above-formulated problem of linear conjugation (for $g(t) \equiv 0$) has, generally speaking, infinitely many linearly independent solutions.

2. By the determinant of the matrix $G(t)$ we shall mean the limit of the sequence of principal minors of the matrix $G(t)$, i.e.

$$\det G(t) = \lim_{k\to\infty} \det G_k(t), \tag{7}$$

under the assumption that the sequence $\{\det G_k(t)\}$, $k = 1, 2, \ldots$, converges uniformly.

Theorem 2. Suppose that the matrix $G(t)$ has a determinant different from zero at each point $t \in C$.

Then there is a representation

$$G(t) = A(t)D(t)B(t), \tag{8}$$

where $A(t)$ and $B(t)$ are, respectively, lower and upper triangular matrices whose left upper corner contains the identity matrix of order n and all whose diagonal elements are equal to one, while $D(t)$ is a quasi-diagonal matrix of the form

$$D(t) = \begin{pmatrix} \begin{matrix} g_{11}(t) \dots g_{1n}(t) \\ \dots \\ g_{n1}(t) \dots g_{nn}(t) \end{matrix} & 0 \\ 0 & \begin{matrix} \dfrac{\det G_{n+1}}{\det G_n} \\ \dfrac{\det G_{n+2}}{\det G_{n+1}} \end{matrix} \\ \dots & \end{pmatrix}, \tag{9}$$

the number n being chosen so that

$$\det G_k(t) \neq 0, \qquad k \geq n,$$

$$\frac{1}{2\pi}\left[\arg \frac{\det G_{n+m+1}(t)}{\det G_{n+m}(t)}\right]_C = 0, \qquad m \geq 0. \tag{10}$$

Consider the following homogeneous problems

$$F^+ = D(t)F^-, \tag{11}$$

$$\Phi^+ = A(t)\Phi^-, \tag{12}$$

$$\psi^+ = B(t)\psi^-. \tag{13}$$

It is easy to see that for the index of problem (11) the formula

$$\varkappa_D(E_p) = \frac{1}{2\pi}[\arg \det D(t)]_C \tag{14}$$

holds.

Further, if the matrices $A(t)$ and $B(t)$ satisfy conditions of type a), b), c) of item 1, then neither problems (12), (13), nor their adjoints have nontrivial solutions vanishing at infinity. Taking into account that $\det A = \det B = 1$, we also obtain

$$\varkappa_A(E_p) = \frac{1}{2\pi}[\arg \det A(t)]_C = 0, \qquad \varkappa_B(E_p) = \frac{1}{2\pi}[\arg \det B(t)]_C = 0. \tag{15}$$

Introduce three singular integral operators

$$K_D f \equiv [I + D(t)]f(t) + \frac{I - D(t)}{\pi i}\int_C \frac{f(\tau)}{\tau - t}\,d\tau, \tag{16}$$

$$K_A \varphi \equiv [I + A(t)]\varphi(t) + \frac{I - A(t)}{\pi i}\int_C \frac{\varphi(\tau)}{\tau - t}\,d\tau, \tag{17}$$

$$K_B\psi \equiv [I + B(t)]\psi(t) + \frac{I - B(t)}{\pi i}\int_C \frac{\psi(\tau)}{\tau - t}\,d\tau, \tag{18}$$

where I is the identity matrix, acting in the space $E_p(C)$. These operators are Noetherian, i.e., bounded, normally solvable, and have finite index in the space $E_p(C)$, with

$$\operatorname{Ind} K_D = \varkappa_D(E_p), \qquad \operatorname{Ind} K_A = \varkappa_A(E_p), \qquad \operatorname{Ind} K_B = \varkappa_B(E_p). \tag{19}$$

Direct computations show that the operator

$$K_G\varphi \equiv [I + G(t)]\varphi(t) + \frac{I - G(t)}{\pi i}\int_C \frac{\varphi(\tau)}{\tau - t}\,d\tau \tag{20}$$

differs from the composition $K_{AK_{DK}}B$ of the singular operators (16), (17), (18) by a completely continuous operator. Consequently, the singular operator K_G is Noetherian, and the index of the operator K_G can be computed by the formula

$$\operatorname{Ind} K_G = \operatorname{Ind} K_A + \operatorname{Ind} K_B + \operatorname{Ind} K_D. \tag{21}$$

3. The results of the preceding sections make it possible to establish the principal assertion of the present paper:

Theorem 3. If $\det G(t) \neq 0$ and the matrices $A(t)$ and $B(t)$ from representation (8) satisfy conditions a), b), c) of Section 1, then

$$\varkappa_G = l - l' = \frac{1}{2\pi}[\arg\det G(t)]_C, \tag{22}$$

where l, l' denote respectively the numbers of linearly independent solutions, vanishing at infinity, of problems (3) (for $g \equiv 0$) and (6).

In order that the nonhomogeneous problem (3) have solutions vanishing at infinity, it is necessary and sufficient that

$$\int_C \sigma^k(t)\,g(t)\,dt = 0, \qquad K = 1, 2, \dots, l', \tag{23}$$

where $\{\sigma^k(t)\}$ is a complete system of linearly independent solutions of the singular integral equation

$$[I + G'(t)]\sigma(t) - \frac{1}{\pi i}\int_C \frac{I - G'(\tau)}{\tau - t}\sigma(\tau)\,d\tau = 0. \tag{24}$$

Remark 1. The fact that we considered a simply connected domain is inessential. All the results of the present paper also remain valid in the case of a multiply connected domain bounded by a finite number of closed curves.

Remark 2. By the method of N. I. Muskhelishvili one can also study a problem of Riemann–Hilbert type for a domain bounded by the unit circle.

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2. N. P. Vekua, *Systems of Singular Integral Equations*, Moscow–Leningrad, 1950.

Note: Figure translations are in progress. See original paper for figures.