A SYMMETRY THEOREM FOR SHALLOW EQUILIBRIUM FORMS OF GEOMETRICALLY NONLINEAR SHALLOW AND NONSHALLOW SHELLS

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Abstract

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A SYMMETRY THEOREM FOR SHALLOW EQUILIBRIUM FORMS OF GEOMETRICALLY NONLINEAR SHALLOW AND NONSHALLOW SHELLS

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1. In paper ([1]) a theorem was proved, called there the symmetry principle, which establishes the necessary and sufficient conditions imposed on the initial form of a shallow geometrically nonlinear shell for the existence, under a given load and prescribed boundary conditions, of equilibrium forms that are pairwise symmetric with respect to its plan. From this theorem there follows a number of qualitative properties of the solution for shells satisfying the symmetry principle. In ([1]) the simplified theory of shallow shells was considered, when they were represented as plates with initial deflection ([2–6]) and others. In the present work the symmetry theorem is generalized to the more exact theory of shells ([3–5, 7–10]) and others, and also to the case of shallow equilibrium forms of nonshallow shells.

2. Let us write the equations of the theory of geometrically nonlinear nonshallow shells for small strains in orthogonal curvilinear coordinates ([4])

$$\left\{\frac{1}{A_1}\left[(A_2\varepsilon_{22})_{,1} - (A_1\varepsilon_{12})_{,2} - \varepsilon_{11}A_{2,1} - \varepsilon_{12}A_{1,2}\right]\right\}_{,1} + \{1, \vec{2}\} =$$

$$= A_1A_2(\chi_{12}^2 - \chi_{11}\chi_{22} - \varkappa_{11}k_{22} - \varkappa_{22}k_{11} + 2\chi_{12}k_{12}); \qquad (1)$$

$$(A_1\varkappa_{11})_{,2} - (A_2\varkappa_{12})_{,1} - \varkappa_{22}A_{1,2} - \varkappa_{12}A_{2,1} + (k_{11} + \chi_{11})(2\varepsilon_{12}A_{2,1} -$$

$$-A_1\varepsilon_{11,2}) + (k_{12} + \chi_{12})[4\varepsilon_{12}A_{1,2} + A_2(\varepsilon_{11} - \varepsilon_{22})_{,1}] +$$

$$+(k_{22}+\chi_{22})[2(A_2\varepsilon_{12})_{,1}-2(\varepsilon_{11}-\varepsilon_{22})A_{1,2}-A_1\varepsilon_{11,2}]=0 \qquad \overleftrightarrow{(1,2)}; \quad (2)$$

$$(A_2T^*_{11})_{,1}+(A_1T^*_{21})_{,2}+T^*_{12}A_{1,2}-T^*_{22}A_{2,1}+$$

$$+A_1A_2(N^*_1k^*_{11}+N^*_2k^*_{12}+X^*_1)=0; \qquad \overleftrightarrow{(1,2)}; \quad (3)$$

$$(A_2N^*_1)_{,1}+(A_1N^*_2)_{,2}-A_1A_2[(T^*_{11}k^*_{11}+T^*_{12}k^*_{12})+\overleftrightarrow{(1,2)}-X^*_3]=0; \quad (4)$$

$$(A_2M^*_{11})_{,1}+(A_1M^*_{21})_{,2}+M^*_{12}A_{1,2}-M^*_{22}A_{2,1}+A_1A_2(L^*_1-N^*_1)=0; \qquad \overleftrightarrow{(1,2)}, \quad (5)$$

where α_1, α_2 are orthogonal coordinates on the undeformed surface; derivatives with respect to them are denoted by indices $(\ )_{,i}=\partial(\ )/\partial\alpha_i$; the sign $\{1,2\}$ or $\overleftrightarrow{(1,2)}$, appearing inside an equation, means that the term denoted by it has the same structure as the preceding one but differs from it by interchanging the indices 1 and 2; $\overleftrightarrow{(1,2)}$ at the end of an equation means that the following equation has the same structure, where the indices 1 and 2 must be interchanged, so that, for example, expressions (2), (3), and (5) consist of two equations; A_1^2, A_2^2 and k_{11}, k_{22}, k_{12} are, respectively, the coefficients of the first quadratic form, the curvatures of the coordinate lines, and the geodesic torsion of the initial surface; ε_{ik} are the compo-

components of the strain tensor of the middle surface; $\chi_{ik}=k^*_{ik}-k_{ik}$—the changes in curvature and torsion of the middle surface as a result of deformation; quantities marked by an asterisk refer to the deformed state; T^*_{ik}, N^*_i, and M^*_{ik} are, respectively, membrane normal and shear forces, transverse forces, bending and twisting moments in projections onto the deformed element of the shell; X^*_i $(i=1,2,3)$ are the projections of the distributed external forces onto the unit vectors of the coordinate lines and the normal to the deformed surface, forming a right-handed coordinate system; L^*_i are the components of the distributed moment of the external forces.

The 3 expressions (1), (2) and the 5 relations (3), (4), (5) are, respectively, the compatibility and equilibrium equations. The boundary conditions will be given below.

If in equations (3) the underlined terms are discarded, then these shortened equations, together with the remaining ones, will determine shallow equilibrium forms of non-shallow shells. If, in addition, the asterisks are discarded in (3), (4), and (5), then we obtain the deformation equations for shallow shells. In what follows we shall work with the shortened equations (3), which will make it possible simultaneously to consider both shallow equilibrium forms of non-shallow shells and shallow shells in general.

3. Let us introduce also a Cartesian coordinate system for the space in which the shell is situated. Consider two equilibrium forms symmetric with respect to the plane xoy (which is drawn so that it does not intersect the shell in the undeformed state). For this pair of symmetric forms the equality holds

$$\overline{B} + \overline{\overline{B}} = -2f, \tag{6}$$

where B is the distance along the z-axis between the deformed and undeformed surfaces; quantities with one overbar denote those referring to the first equilibrium form, and quantities with two overbars to the second, symmetric to it; f is the distance of points of the undeformed (initial) surface of the shell from the plane xoy.

Let us find the necessary conditions for the existence of forms satisfying condition (6). It is obvious (if, for example, the deformed surface is represented in the form $z = f + B$) that when (6) holds, for both symmetric surfaces the coefficients of the first quadratic forms coincide; the coefficients of the second quadratic forms are equal in magnitude and opposite in sign; the horizontal projections of the displacement vectors at symmetric points are equal. Hence

$$\overline{k}^*_{ij} = \overline{\overline{k}}^*_{ij}; \qquad \overline{\chi}_{ij} + \overline{\overline{\chi}}_{ij} = -2k_{ij}; \qquad \overline{\varepsilon}_{ij} = \overline{\overline{\varepsilon}}_{ij} \quad (i, j = 1, 2). \tag{7}$$

From (7) follows (8), if, for relating the forces and moments to the deformation components of the middle surface, one applies Love' s formulas for small deformations

$$\overline{T}^*_{ij} = \overline{\overline{T}}^*_{ij}; \qquad \overline{M}^*_{11} + \overline{\overline{M}}^*_{11} = -2D\left(k_{11} + \mu k_{22}\right); \qquad (\overrightarrow{1,2})(i, j = 1, 2);$$

$$\overline{M}^*_{12} + \overline{\overline{M}}^*_{12} = \overline{M}^*_{21} + \overline{\overline{M}}^*_{21} = -2D(1-\mu)k_{12}. \tag{8}$$

From (7) and (8) it immediately follows that the compatibility equations and the first two shortened equations (3) for both symmetric equilibrium forms coincide, if the Gauss-Codazzi identities for the undeformed surface are taken into account and under the condition that $\overline{X}^*_i = \overline{\overline{X}}^*_i$ $(i = 1, 2)$. Further, adding equations (4) and (5) for both states, we obtain the relations

$$\left[A_2\left(\overline{N}^*_1 + \overline{\overline{N}}^*_1\right)\right]_{,1} + [\overrightarrow{1,2}] + A_1A_2\left(\overline{X}^*_3 + \overline{\overline{X}}^*_3\right) = 0; \tag{9}$$

$$[A_2(\overline{M}^*_{11}+\overline{\overline{M}}^*_{11})]_{,1}+[A_1(\overline{M}^*_{21}+\overline{\overline{M}}^*_{21})]_{,2}+A_{1,2}(\overline{M}^*_{12}+\overline{\overline{M}}^*_{12})-$$

$$-A_{2,1}(\overline{M}^*_{22}+\overline{\overline{M}}^*_{22})+A_1A_2[(\overline{L}^*_1+\overline{\overline{L}}^*_1)-(\overline{N}^*_1+\overline{\overline{N}}^*_1)]=0; \qquad \binom{1,2}{\longleftrightarrow}. \quad (10)$$

If from (9) and (10) one eliminates $(\overline{N}^*_i+\overline{\overline{N}}^*_i)$ $(i=1,2)$ and then replaces $\overline{M}^*_{ik}+\overline{\overline{M}}^*_{ik}$ by k_{ij} by means of (8), then the first desired necessary condition (11) is obtained, which we simplify using the Gauss-Codazzi relations:

$$\left\{\frac{A_2}{A_1}(k_{11}+k_{22})_{,1}\right\}_{,1}+\left\{\binom{1,2}{\longleftrightarrow}\right\}_{,2}=\frac{A_1A_2}{2D}(\overline{X}^*_3+\overline{\overline{X}}^*_3)+$$

$$+\frac{1}{D}\left\{[A_2(\overline{L}^*_1+\overline{\overline{L}}^*_1)]_{,1}+\left[\binom{1,2}{\longleftrightarrow}\right]\right\}. \quad (11)$$

Equality (11) in the general case is a nonlinear differential equation of the 4th order for the function f, determining the initial form of the shell. It is also necessary to prescribe two boundary conditions on the contour C of the initial surface, which are obtained from the boundary conditions of the problem, using (6), (7), and (8). Thus, for example, in the case of a hinged immovable edge:

$$f\big|_C=0; \qquad -2D\left[(k_{11}+\mu k_{22})n_1^2+\binom{1,2}{\longleftrightarrow}n_2^2+(1-\mu)k_{12}n_1n_2\right]\Bigg|_C=\overline{G}^*+\overline{\overline{G}}^* \quad (12)$$

where n_i $(i=1,2)$ are the projections of the unit normal $\mathbf{n}$ to the contour C, lying in the tangent plane to the initial form of the shell at the points C; G^* is the prescribed bending moment on the contour. For rigid clamping we have

$$f\big|_C=0; \qquad \partial f/\partial n\big|_C=0. \quad (13)$$

If the edge is not fixed, then, in addition to the second condition (12), it is necessary to prescribe also the following:

$$-\frac{2D}{A_1A_2}\left\{n_1\left[A_2(k_{11}+k_{22})_{,1}+(1-\mu)k_{11}(A_{2,1}-A_{1,2})\right]+n_2\left[\binom{1,2}{\longleftrightarrow}\right]\right\}+$$

$$+2D(1-\mu)\frac{\partial}{\partial s_C}\left[n_1n_2(k_{11}-k_{22})-(n_1^2-n_2^2)\right]\Bigg|_C=(\overline{N}^*+\overline{\overline{N}}^*)-\frac{\partial}{\partial s_C}(\overline{H}^*+\overline{\overline{H}}^*), \quad (14)$$

where N^*, H^* are, respectively, the prescribed contour transverse forces and twisting moments, etc.

Thus, if there exist solutions satisfying (6) for $\overline{X}_i^* = \overline{\overline{X}}_i^*$ $(i = 1, 2)$, then it is necessary that $f(a_1, a_2)$ satisfy equation (11) with the corresponding boundary conditions given above.

The sufficiency of these conditions is proved in the usual way, i.e., by reversing the proof of necessity.

Thus, the following theorem has been established:

Symmetry theorem. *The differential equation (11) and the corresponding boundary conditions (see (12), (13), (14), etc.) for the function f, determining the form of the initial surface of the shell, are necessary and sufficient conditions for the existence of pairwise symmetric equilibrium forms satisfying the equations of the problem, provided that* $\overline{X}_i^* = \overline{\overline{X}}_i^*$ $(i = 1, 2)$. *These symmetric forms may arise through variation of the parameters of the transverse load* X_3^**, the contour bending moments, and the transverse forces.*

From this theorem, as a special case, there follows the symmetry principle of [1].

4. Shells in which the conditions of the theorem are satisfied possess the following basic properties, the same as in ([1]). (Let us fix, for both symmetric forms, all loads except the transverse one, which, as shown in ([1]), does not diminish the generality of the reasoning.)

a) The characteristic of the shell (diagram—the load parameter $q = X_3^*(a_1, a_2)$ —the deflection rise $V = B(a_1, a_2)$; a_1, a_2 are the coordinates of an arbitrarily chosen unfixed point) is a skew-symmetric curve. This follows from the fact that for any pair of symmetric forms the relation (15), following from (6) and (11), holds:

$$\bar{q} + \bar{\bar{q}} = 2k_1; \qquad \bar{V} + \bar{\bar{V}} = 2k_2, \tag{15}$$

where k_i are constants for the given shell, determined in solving the corresponding boundary-value problem for equation (11). The center of symmetry has coordinates k_1 and k_2.

b) In view of the skew-symmetry of the characteristic, the first equality (15) gives a relation between the upper and lower critical loads. For the same reason, k_1 (the ordinate of the center of symmetry) is a lower bound for the values of the upper critical load and an upper bound for the lower one.

c) For $\bar{q} = q = k_1$, we have $\bar{B} \equiv \bar{\bar{B}} \equiv -f$ (see (6)), i.e., the shell straightens. If all longitudinal loads are absent, then for $\bar{\bar{q}} = 2k_1$ (i.e., $\bar{q} = 0$ and $\bar{B} = 0$), $\bar{\bar{B}} = -2f$—the shell turns completely inside out. These phenomena can occur only when the symmetry theorem is valid.

d) Suppose, in addition, that some transverse force factor also acts (among such factors may be any of those entering the boundary conditions for

f) with parameter η. Let us consider the symmetric forms generated by the loads $\bar{q}, +\eta$ and $\bar{\bar{q}}, -\eta$. We shall construct the characteristics in the same plane (V, q) for fixed $\eta \neq 0$. Each of these curves will no longer be skew-symmetric (it is skew-symmetric only for $\eta = 0$), but the characteristics for $+\eta$ and $-\eta$ will be mutually symmetric with respect to the same center of symmetry as for $\eta = 0$. A pencil of symmetric characteristics is obtained. This property follows from the fact that the terms containing the parameters $+\eta$ and $-\eta$ cancel each other in the derivation of equations (9), (10), (11) and of conditions (12), (14).

Just as for shallow, simplest shells ([1]), one can determine the exact value of the rise of the initial deflection beginning from which snap-through appears; introduce the concept of nearly symmetric systems, etc.

Let us note that if, in E. Reissner' s equations (11) for asymmetric deformations of non-shallow shells, one abandons the simplifications introduced into (11), then the corresponding symmetry theorem holds.

In ([12]) it is shown how, using the property of symmetric pencils, one can substantially simplify the construction of the solution.

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Note: Figure translations are in progress. See original paper for figures.

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