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Abstract

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THEORY OF ELASTICITY

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ON AVERAGING IN SYSTEMS OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

Integro-differential equations were first studied by the averaging method ([1]) in works ([3–6]), where it was shown that, in the general case, to one system of integro-differential equations there can be associated (in contrast to averaging in differential equations) three different systems of averaged equations (some of these systems will be differential, others integro-differential).

In the present article one of the variants of averaging is considered (however, everything said below also applies to the other variants of averaging). Averaging theorems are proved both on a finite and on an infinite interval. Systems with slow variables and systems with fast and slow variables are considered. In particular, when the integral term is absent in (1), the theorem of N. N. Bogolyubov ([1]) is obtained from Theorem 1 below, under weaker restrictions.

Systems of integro-differential equations of the type (1), (7) lead to problems in the dynamics of imperfectly elastic bodies that do not have sufficiently effective methods of solution. Indeed, the relations between the stresses $\sigma_{ij}(x,t)$ and the displacement vector $u(x,t)$ for such media have, for example, the form ([2])

$$\sigma_{ij} - \sigma\delta_{ij} = f_0(t, z(t,t))\varepsilon_{ij}(x,t) - \varepsilon\int_0^t f_1(t-\tau, z(\tau,\tau))\varepsilon_{ij}(x,\tau)\,d\tau,$$

where $\sigma = K\operatorname{div} u$, $2\varepsilon_{ij} = u_{i,j}+u_{j,i}$, $z(t,t) = \varepsilon_{mn}(x,t)\varepsilon_{mn}(x,t)$ (for imperfectly elastic systems ε is a small parameter). With a suitable choice of the functions $\psi_{ik}(x)$, $u_i = \psi_{ik}(x)\varphi_{ik}(t)$, and from the equations of motion $\rho\ddot{u}_i = \sigma_{ij,j}$, by the Bubnov–Galerkin method we obtain for $\varphi_{ik}(t)$ a system of equations easily reducible to the form (7) (here k is a vector index).

In what follows, the notation $f(t,x) \in \operatorname{Lip}_x(\Omega, \lambda(t))$ means that $f(t,x)$ satisfies in the domain Ω the Lipschitz condition with respect to x with Lipschitz function $\lambda(t)$.

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1. Let there be given a system (x and φ are vectors, $\varepsilon > 0$ is a small parameter)

$$\dot{x} = \varepsilon X\left(t, x, \int_0^t \varphi(t, s, x(s))\, ds\right). \tag{1}$$

Theorem 1. *Let the functions $X(t, x, y)$ and $\varphi(t, s, x)$, defined and continuous in the domain Ω ($t \geq 0,\ s \geq 0,\ x \in D \subset E_n,\ y \in E_m$), satisfy the following conditions:*

1)
$$X(t, x, y) \in \mathrm{Lip}_{x,y}(\Omega, \lambda), \quad \varphi(t, s, x) \in \mathrm{Lip}_x(\Omega, \mu(t, s));$$

2)
$$\int_0^t d\tau \int_0^\tau \mu(\tau, s)\, ds \leq c \quad (c = \mathrm{const},\ \lambda = \mathrm{const}),$$

3) *at each point x of the domain D there exists the limit*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X(t, x, \psi(t, x))\, dt = X_0(x), \qquad |X_0(x)| \leq M,$$

$$X_0(x) \in \mathrm{Lip}_x(D, \nu) \qquad \psi(t, x) = \int_0^t \varphi(t, s, x)\, ds, \qquad \nu = \mathrm{const}, \quad M = \mathrm{const},$$

4) the solution $\xi = \xi(t),\ \xi(0) = x(0)$ of the averaged equation

$$\dot{\xi} = \varepsilon X_0(\xi) \tag{2}$$

is defined for all $t \geqslant 0$ and lies in the domain D together with its ρ-neighborhood.

Then for any $\eta > 0$ and $L > 0$ one can indicate such an ε_0 that, for $\varepsilon < \varepsilon_0$, on the interval $0 \leqslant t \leqslant L\varepsilon^{-1}$ the inequality

$$|x(t) - \xi(t)| < \eta \tag{3}$$

will hold.

Proof. Representing systems (1) and (2) in the form of integral equations, we find

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$$x-\xi=\varepsilon\int_0^t\left[X\left(\tau,x,\int_0^\tau\varphi(\tau,s,x(s))\,ds\right)-X\left(\tau,\xi,\int_0^\tau\varphi(\tau,s,\xi(s))\,ds\right)\right]d\tau$$
$$+\int_0^t\left[X\left(\tau,\xi,\int_0^\tau\varphi(\tau,s,\xi(s))\,ds\right)-X_0(\xi(\tau))\right]d\tau. \tag{4}$$

As in (7), it can be shown that the second term in (4), for sufficiently small ε, on the interval $0\leqslant t\leqslant L\varepsilon^{-1}$, will be less than any prescribed number δ. Now, using the known inequality (8), from (4) we find

$$|x-\xi|<\delta\exp\left[\lambda\varepsilon t+\varepsilon\lambda\int_0^t d\tau\int_0^\tau\mu(\tau,s)\,ds\right]\leqslant\delta\exp(\lambda L+\varepsilon\lambda c).$$

Putting $\delta=\min(\rho,\eta)\exp(-\lambda L-\varepsilon\lambda c)$, we obtain the assertion of the theorem.

Remark 1. If in the formulation of the theorem condition 2) is omitted and μ is regarded as constant, then inequality (3) will hold on the interval $0\leqslant t\leqslant L\varepsilon^{-1/2}*$.

Remark 2. Theorem 1 is obviously also extended to systems of equations of the form

$$\dot{x}=\varepsilon X(t,x(t),\dot{x}(t),x(t-\Delta),\dot{x}(t-\Delta),$$

$$\int_0^t\varphi(t,s,x(s),\dot{x}(s),x(s-\Delta),\dot{x}(s-\Delta))\,ds,\varepsilon). \tag{5}$$

Theorem 2. Let the functions $X(t,x,y)$ and $\varphi(t,s,x)$ satisfy conditions 1) and 2) of Theorem 1 and be such that system (1) has no singular points (8), and let:

3) at each point $x\in D$, uniformly with respect to t, there exists the limit

$$\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}X(\tau,x,\psi(\tau,x))\,d\tau=:X_0(x),\qquad X_0(x)\in\mathrm{Lip}_x(D,\nu), \tag{6}$$

$$|X_0|\leqslant M,\quad \nu=\text{const};$$

4) the solution $\xi=\xi(t)$, $\xi(0)=x(0)$ of the averaged system (2) is defined for all $t\geqslant 0$ and lies in the domain D together with its ρ-neighborhood, and the limiting passage in (6) is performed uniformly with respect to $x\in S_\rho\subset D$ (S_ρ is the ρ-neighborhood of the solution $\xi(t)$).

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* In article (3) the following corrections must be made: 1) in the formulation of the theorem, where it says $0 < k < 1/2$, one should read $0 < k \leqslant 1/2$; 2) in formula (4), instead of the term $4Net$ there should be the term $2a/\varepsilon$.

5) the solution $\xi = \xi(\tau)$, $\tau = \varepsilon t$, of the averaged system is asymptotically stable with respect to τ.

Then for any $0 < \eta < \rho$ one can specify an ε_0 such that, for $\varepsilon < \varepsilon_0$, for all $t \geqslant 0$ the inequality

$$|x(t) - \xi(t)| < \eta$$

will hold.

Proof is carried out with the aid of the methods of papers ([7, 9]) and using Theorem 1.

Remark 3. Theorem 2 also extends to systems of the form (5).

Remark 4. If system (1) is written in the form

$$\dot{x} = \varepsilon X\left(t, x, y, \int_0^t \varphi_1(t, s, x(s), y(s))\, ds\right),$$

$$\dot{y} = \varepsilon Y\left(t, x, y, \int_0^t \varphi_2(t, s, x(s), y(s))\, ds\right),$$

then in it one can carry out **partial** averaging, averaging, for example, only the first group of equations (the average of Y may not even exist):

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X(t, x, y, f(t, x, y))\, dt = X_0(x, y),$$

$$f(t, x, y) = \int_0^t \varphi_1(t, s, x, y)\, ds.$$

The corresponding partially averaged system will have the form

$$\dot{\xi} = \varepsilon X_0(\xi, \eta), \qquad \dot{\eta} = \varepsilon Y\left(t, \xi, \eta, \int_0^t \varphi_2(t, s, \xi(s), \eta(s))\, ds\right).$$

Theorems 1 and 2 also extend to the case of partial averaging.

2. Let us now consider systems with fast and slow variables

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$$\dot{x} = \varepsilon X\left(t, x, y, \dot{x}, \dot{y}, \int_0^t \varphi(t, s, x(s), y(s), \dot{x}(s), \dot{y}(s))\, ds, \varepsilon\right),$$

$$\dot{y} = Y_0(t, x, y) + \varepsilon Y_1\left(t, x, y, \dot{x}, \dot{y}, \int_0^t \varphi_1(t, s, x(s), y(s), \dot{x}(s), \dot{y}(s))\, ds, \varepsilon\right). \quad (7)$$

Suppose that the general solution $y = F(t, x, c)$ of the degenerate (5) system

$$\dot{y} = Y_0(t, x, y); \qquad x = \text{const},$$

is known. Then system (7) is reduced to the form

$$\dot{x} = \varepsilon X\left(t, x, F, \dot{F}, \int_0^t \varphi(t, s, x(s), \dot{x}(s), F, \dot{F})\, ds, \varepsilon\right),$$

$$\dot{c} = \varepsilon\left(\frac{\partial F}{\partial c}\right)^{-1}\left[-\frac{\partial F}{\partial x}X + Y_1\right]. \quad (8)$$

System (8) is a system of type (1). Therefore Theorems 1 and 2 extend to it. Further, partial averaging can be applied to system (8), averaging, for example, the first group of equations in (8). In this case we obtain averaging theorems for systems of the form (7) as for

finite as well as on infinite intervals. These theorems are analogous to Theorems 1 and 2.

Systems of the form (7) with a retarded argument are considered in an analogous way.

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