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# ON THE CENTRAL LIMIT THEOREM

MATHEMATICS

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**Abstract**

**Full Text**

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**MATHEMATICS**

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## **ON THE CENTRAL LIMIT THEOREM FOR STATIONARY PROCESSES**

*(Presented by Academician Yu. V. Linnik on 26 II 1969)*

In the present note it is shown that the central limit theorem (c.l.t.) for certain classes of stationary, in the narrow sense, random processes can be obtained from the c.l.t. for stationary sequences of martingale differences.

Suppose that a space  $X$  with a  $\sigma$ -algebra of sets  $M$  and a probability measure  $P$  is given. The spaces  $\mathbf{L}_p$  correspond to the measure  $P$ ;  $|f|_p$  is the norm of the function  $f$  in  $\mathbf{L}_p$ .

If a  $\sigma$ -algebra  $L$  is contained in  $M$ , then  $H(L)$  denotes the Hilbert space of those functions from  $\mathbf{L}_2$  that are measurable with respect to  $L$ .

The symbol  $P_G$  denotes the orthoprojector onto the closed subspace  $G \subset H = \mathbf{L}_2$ .

Let  $T$  be an ergodic automorphism of the space  $X$  with measure  $P$ , and let  $M_0$  be a  $\sigma$ -algebra such that  $T^{-1}(M_0) \subset M_0$ . The relation  $Uf(x) = f(Tx)$  defines a unitary operator in  $H$ .

Finally, set  $M_k = T^{-k}(M_0)$ ,  $H_k = H(M_k) = U_0^{kH}$ ,  $S_k = H_k \ominus H_{k+1}$ . By  $R$  we denote the linear space of those elements  $g \in H$  such that  $g \in H_k \ominus H_l$  for some  $k$  and  $l$ ,  $-\infty < k \leq l < \infty$ .

**Theorem 1.** Let  $f \in \mathbf{L}_2$  and

$$\inf_{g \in R} \lim_{n \rightarrow \infty} \frac{\left| \sum_{k=0}^{n-1} U^k(f - g) \right|_2}{\sqrt{n}} = 0.$$

Then there exists

$$\lim_{n \rightarrow \infty} \left| \sum_{k=0}^{n-1} U^k f \right|_2 / \sqrt{n} = \sigma \tag{1}$$

and

$$\mathbf{P} \left\{ \sum_{k=0}^{n-1} U^k f / \sqrt{n} < z \right\} \xrightarrow{n \rightarrow \infty} \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^z e^{-u^2/2\sigma^2} du. \quad (2)$$

We outline the proof of the theorem. Let  $\varepsilon_p > 0$ ,  $\varepsilon_p \xrightarrow{p \rightarrow \infty} 0$ ,  $f_p \in R$ , and

$$\overline{\lim}_{n \rightarrow \infty} \left| \sum_{k=0}^{n-1} U^k (f - f_p) \right|_2 / \sqrt{n} < \varepsilon_p.$$

Consider the chain of equalities

$$\begin{aligned} f &= f_p + f - f_p = \sum_{l=-\infty}^{\infty} P_{S_l} f_p + f - f_p = \sum_{l=-\infty}^{\infty} U^{-l} P_{S_l} f_p + \\ &+ \sum_{l=-\infty}^{\infty} \sum_{m=0}^{-l-1} U_{S_l}^{mP} f_p - U \sum_{l=0}^{\infty} \sum_{m=0}^{-l-1} U_{S_l}^{mP} f_p + f - f_p = h_p + g_p - U g_p + f - f_p. \end{aligned}$$

Further,

$$\begin{aligned} &\overline{\lim}_{n \rightarrow \infty} \left| \sum_{k=0}^{n-1} U^k (f - h_p) \right|_2 / \sqrt{n} \leq \\ &\leq \overline{\lim}_{n \rightarrow \infty} \left\{ |g_p - U^n g_p|_2 + \left| \sum_{k=0}^{n-1} U^k (f - f_p) \right|_2 \right\} / \sqrt{n} < \varepsilon_p. \quad (3) \end{aligned}$$

Let us now note that

$$h_p = \sum_{l=-\infty}^{\infty} U^{-l} P_{S_l} f_p \in S_0, \quad U^k h_p \in S_k.$$

Hence it follows that  $U^k h_p$  is measurable with respect to  $M_k$  and is orthogonal to  $H_{k+1} = H(M_{k+1})$ , i.e., the sequence  $U^{-k} h_p$  is an ergodic sequence of martingale differences. Therefore (see (1, 2)) the quantities

$$\sum_{k=0}^{n-1} U^k h_p / \sqrt{n}$$

are asymptotically normally distributed with variance  $\sigma_p^2 = |h_p|^2$ .

The sequence  $\sigma_p$  converges to some limit  $\sigma$ , since

$$\begin{aligned} |\sigma_p - \sigma_{p'}| &\leq |h_p - h_{p'}|_2 = \left| \sum_{k=0}^{n-1} U^k(h_p - h_{p'}) \right|_2 / \sqrt{n} \leq \\ &\leq \overline{\lim}_{n \rightarrow \infty} \left\{ \left| \sum_{k=0}^{n-1} U^k(f - h_p) \right|_2 + \left| \sum_{k=0}^{n-1} U^k(f - h_{p'}) \right|_2 \right\} / \sqrt{n} \leq \varepsilon_p + \varepsilon_{p'}. \end{aligned}$$

Relation (2) now follows from Lemma 5.3 of (4); equality (1) follows from (3) and from the fact that  $\sigma_p \rightarrow \sigma$ .

**Theorem 2.** *Let  $T$  be an ergodic automorphism,  $M_0$  the same  $\sigma$ -algebra as in Theorem 1;  $f \in \mathbf{L}_{2+\delta}$  for some  $0 \leq \delta \leq \infty$  and*

$$\sum_{A \geq 0} (|P_{H_A} f|_{(2-\delta)/(1+\delta)} + |f - P_{H_{-A}} f|_{(2+\delta)/(1+\delta)}) < \infty.$$

*Then the condition of Theorem 1 is satisfied.*

Let us explain how Theorem 2 is proved. Instead of  $\int_X g(x)h(x)\mathbf{P}(dx)$  we shall write  $(g, h)$ . Put

$$f_1^{(A)} = P_{H_A} f, \quad f_2^{(A)} = f - P_{H_{-A}} f, \quad r_i^{(A)}(k) = (f_i^{(A)}, U^k f_i^{(A)}) \quad (i = 1, 2).$$

From Hölder' s inequality and the well-known property of conditional mathematical expectations (see, for example, (3), p. 508) it follows that

$$\begin{aligned} |r_i^{(A)}(k)| &= |(f_i^{(A)}, U^{|k|} f_i^{(A)})| = |(f_i^{(A)}, U^{\pm|k|} f_i^{(A+|k|)})| \leq \\ &\leq |f_i^{(A)}|_{2+\delta} |f_i^{(A+|k|)}|_{(2-\delta)/(1+\delta)} \leq 2|f|_{2+\delta} |f_i^{(A+|k|)}|_{(2+\delta)/(1+\delta)}. \end{aligned}$$

Using this estimate, we find that

$$\begin{aligned} &\overline{\lim}_{n \rightarrow \infty} \left| \sum_{k=0}^{n-1} U^k(f_1^{(A)} + f_2^{(A)}) \right|_2 / \sqrt{n} \leq \sum_{i=1}^2 \overline{\lim}_{n \rightarrow \infty} \left| \sum_{k=1}^{n-1} U^k f_i^{(A)} \right|_2 / \sqrt{n} = \\ &= \sum_{i=1}^2 \lim_{n \rightarrow \infty} \left( \sum_{|k| < n} \left( 1 - \frac{|k|}{n} \right) r_i^{(A)}(k) \right)^{1/2} \leq 2|f|_{2+\delta}^{1/2} \sum_{i=1}^2 \left( \sum_{k=A}^{\infty} |f_i^{(k)}|_{(2+\delta)/(1+\delta)} \right)^{1/2} \rightarrow 0, \quad A \rightarrow \infty. \end{aligned}$$

It remains to note that

$$f - f_1^{(A)} - f_2^{(A)} \in R.$$

**Remark 1.** Theorems 1 and 2 can be reformulated for the case where  $T$  is an endomorphism. In this case the spaces  $S_k$  form a sequence infinite in one direction.

**Remark 2.** Under the conditions of Theorem 2, the equality  $f = g + Uh - h$  holds, where  $g, h \in \mathbf{L}^{(2+\delta)/(1+\delta)}$ ,  $g$  is measurable with respect to  $M_0$  and has zero integrals over all sets from  $M_1$ . Such a representation for  $\delta = 0$  is useful in the proof of more refined limit theorems, for example, in the proof of weak convergence of distributions in the space of continuous functions corresponding to the sequence of random polygonal lines constructed from the sums

$$\sum_{k=0}^{n-1} U^k f$$

to a measure corresponding to Brownian motion.

**Remark 3.** From Theorem 2 it is not difficult to obtain a number of theorems concerning processes with strong and uniformly strong mixing and functionals of such processes.

In particular, Theorems 18.6.1, 18.6.2, and 18.6.3 from the book <sup>(3)</sup> are consequences of Theorem 2. Moreover, under the conditions of Theorem 18.6.1 and with a certain strengthening of the conditions of Theorems 18.6.2 and 18.6.3 from <sup>(3)</sup>, one can obtain the representation  $f = g + Uh - h$ ,  $g \in S_0$ ,  $h \in \mathbf{L}_2$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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