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MATHEMATICS

1969

SovietRxiv

Abstract

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UDC 513.88:513.83

MATHEMATICS

V. A. GEILER

ON FUNCTORS DEFINED BY REFLEXIVE K-SPACES

(Presented by Academician L. V. Kantorovich on 24 I 1969)

Functors on the category of locally convex spaces were studied by G. H. Berman ([1,2]). He constructed a duality theory for functors on the category $\mathcal{FN}$ of nuclear Fréchet spaces. In particular, he studied the functor $\Lambda_{\mathfrak{M}}$, defined by the perfect space of sequences λ, which, from the point of view of the theory of K-spaces, is a discrete reflexive K-space of countable type ([3,4]). In the present note, for each reflexive K-space X, endowed with a certain locally convex topology, a functor $\Phi_X^{\mathfrak{M}}$ is constructed, of which $\Lambda_{\mathfrak{M}}$ is a special case. We shall also consider a certain functor $\Omega_A^{\mathfrak{S}}$, and it turns out that on certain categories (containing $\mathcal{FN}$) functors of this type are isomorphic to Ω_A, Σ_A, $\Phi_X^{\mathfrak{M}}$, and also to the ε_A introduced in ([5]).

In what follows X denotes a reflexive K-space, $\bar{X}$ its conjugate, $\mathfrak{M}$ a certain system of bounded absolutely convex normal subsets of $\bar{X}$ covering $\bar{X}$ (and called a topologizing system for X); if all the sets in $\mathfrak{M}$ are $\sigma(\bar{X}, X)$-bicompact, then $\mathfrak{M}$ is called bicompact. By $X_{\mathfrak{M}}$ we denote the space X, endowed with the locally convex topology generated by the polars of the sets in $\mathfrak{M}$. Using the results of ([6]), it is not difficult to show that among the spaces obtained in this way there is X_τ—the space X endowed with the Mackey topology $\tau(X, \bar{X})$; we also note that a subset of $X_{\mathfrak{M}}$ is bounded if and only if it belongs to the system $\mathfrak{B}$ of $\sigma(X, \bar{X})$-bounded subsets of X. Using the results of A. G. Pinsker (([4]), Chap. IV, Theorems 1.31, 4.13), we realize X as the base of a K-space $L^1_{\mathrm{loc}}(T;\mu)$, where T is a locally bicompact space which is a sum of extremally disconnected bicompacts; μ is a positive measure on T such that the families of locally μ-negligible ([7]) and nowhere dense subsets of T coincide, and $L^1_{\mathrm{loc}}(T;\mu)$ is the quotient space of the space $\mathcal{L}^1_{\mathrm{loc}}(T;\mu)$ of locally summable functions by the subspace of locally μ-negligible functions. By θ we shall denote the canonical mapping of $\mathcal{L}^1_{\mathrm{loc}}(T;\mu)$ onto $L^1_{\mathrm{loc}}(T;\mu)$.

Let E denote a separable locally convex space, and $\mathfrak{U}$ the family of its absolutely convex neighborhoods of zero. By $\mathcal{X}_{\mathfrak{M}}(E)$ (respectively $\mathcal{X}_{\mathfrak{M}}[E]$) we denote the space of μ-measurable functions from T into E such that for every continuous seminorm p on E, $\theta(pf) \in X$ (for every $a' \in E'$, $\theta(a'f) \in X$), endowed with the topology generated by the seminorms

$$f \mapsto p_{M^0}(\theta(p_U f))$$

(respectively

$$f \mapsto \sup_{a' \in U^0} p_{M^0}(\theta(a' f)),$$

where $M \in \mathfrak{M}$, $U \in \mathfrak{U}$. These spaces do not depend on the choice of the realization of X. By $X_{\mathfrak{M}}(E)$ (respectively $X_{\mathfrak{M}}[E]$) we denote the separable locally convex space associated with $\mathcal{X}_{\mathfrak{M}}(E)$ ($\mathcal{X}_{\mathfrak{M}}[E]$).

Theorem 1. If E is a nuclear space, then $X_{\mathfrak{M}}(E)$ is isomorphic to $X_{\mathfrak{M}}[E]$.

If E, F are locally convex spaces and $\mathfrak{S}$ is a system of bounded subsets of E, then $L^{\mathfrak{S}}(E, F)$ is the space of continuous li-

linear maps from E to F, endowed with the $\mathfrak{S}$-topology; by $\mathfrak{B}$ we shall denote the system of all bounded subsets of E, and by $\mathfrak{E}$ that of all equicontinuous subsets of E'.

Theorem 2. *Let E be a complete nuclear space, and let $\mathfrak{M}$ be a bicompact topologizing system for $\bar{X}$.*

Each of the following conditions is sufficient for $X_{\mathfrak{M}}(E)$ to be isomorphic to $L_{\mathfrak{M}}(X_{\mathfrak{M}}, E)$: a) E is dually nuclear ([8]); b) let $\mathfrak{a}$ be the character of E; if $\mathfrak{F}$ is a collection of nowhere dense subsets T and $\operatorname{card}(\mathfrak{F}) \leq \mathfrak{a}$, then $\bigcup \mathfrak{F}$ is nowhere dense.

Corollary. *If, under the hypotheses of Theorem 2, the system $\mathfrak{M}$ is bicompact, then*

$$X_{\mathfrak{M}}(E) \simeq X_{\mathfrak{M}} \hat{\otimes} E.$$

In what follows, $\mathcal{L}$ denotes the category of separated locally convex spaces, $\mathcal{EL}$ the full subcategory of complete spaces in $\mathcal{L}$, and $\mathcal{IT}$ the category of infrabarreled spaces; we shall also consider the following full subcategories of $\mathcal{EL}$: $\mathcal{BLT}$, of barrelled spaces; $\mathcal{N}$, of nuclear spaces; $\mathcal{DN}$, of nuclear dually nuclear spaces; $\mathcal{RN}$, of reflexive nuclear spaces; and $\mathcal{RDN}$, of reflexive spaces in $\mathcal{DN}$.

For each space $X_{\mathfrak{M}}$ we define a functor $\Phi_X^{\mathfrak{M}}$ on $\mathcal{L}$ as follows: $\Phi_X^{\mathfrak{M}} E = X_{\mathfrak{M}}(E)$, and for $\alpha \in L(E, F)$, $\Phi_X^{\mathfrak{M}} \alpha$ is the mapping induced by passage to quotient spaces from the mapping $f \mapsto \alpha f$ from $\mathcal{X}_{\mathfrak{M}}(E)$ to $\mathcal{X}_{\mathfrak{M}}(F)$. The functors Ω_A and Σ_A are the same as in (1). Let $A \in \mathcal{L}$; $\mathfrak{S}$ is a system of bounded subsets of A covering A; define the functor $\Omega_A^{\mathfrak{S}}$ on the category $\mathcal{K}$ by

$$\Omega_A^{\mathfrak{S}} E = L_{\mathfrak{S}}(A, E), \qquad \Omega_A^{\mathfrak{S}} \alpha = \Omega_A \alpha \quad (\alpha \in L(E, F)).$$

The functor $\Omega_{A_c}^{\mathfrak{E}}$ will also be denoted by Ψ_A.

The following assertions are valid:

a) if $A \in \mathcal{BL}$, then Σ_A is isomorphic to Ψ_A on $\mathcal{N}$; if $A \in \mathcal{N}$, then Σ_A is isomorphic to Ψ_A on $\mathcal{BL}$;

b) for every $A \in \mathcal{L}$, ε_A is isomorphic to $\Omega^{\mathfrak{C}}_{A_c}$ on $\mathcal{L}$;

c) on $\mathcal{DN}$, $\Phi^{\mathfrak{M}}_X$ is isomorphic to $\Omega^{\mathfrak{M}}_{X_\tau}$.

Proof. Assertion a) follows from Grothendieck's theorem ([9], Ch. II, Theorem 6); b) from Corollary 2 to Proposition 4 ([10]); c) from Theorem 2.

If R and S are such functors that the class of mappings from R to S is a set, then by $H_b(R, S)$ we shall denote the space of mappings from R to S, endowed with the initial topology with respect to the family of mappings

$$P_E : H(R, S) \to L_{\mathfrak{B}}(RE, SE). \tag{1}$$

A functor R is called continuous on the category $\mathcal{K}$ if, for every pair of spaces E, F from $\mathcal{K}$, the mapping

$$R : L_{\mathfrak{B}}(E, F) \to L_{\mathfrak{B}}(RE, RF)$$

is continuous; R is called regular on $\mathcal{K}$ if $H_b(\Omega_A, R)$ is canonically isomorphic to the space RA for every A in $\mathcal{K}$ ([1]).

Theorem 3. *For every $A \in \mathcal{L}$, the functor $\Omega^{\mathfrak{S}}_A$ is continuous on $\mathcal{L}$ and regular on $F\mathcal{T}$.*

Theorem 4. *Let $A, B \in \mathcal{L}$; let $\mathfrak{S}$ (respectively $\mathfrak{T}$) be a system of absolutely convex weakly bicompact subsets of A (respectively of B).*

If for every $u \in L(A'_{\mathfrak{S}}, B'_{\mathfrak{T}})$ the adjoint mapping $u' \in L(B, A)$ (in particular, if B is a Mackey space), then

$$H_b(\Omega^{\mathfrak{S}}_A, \Omega^{\mathfrak{T}}_B)$$

is isomorphic to

$$L_{\mathfrak{B}}(A'_{\mathfrak{S}}, B'_{\mathfrak{T}}).$$

By $\mathcal{D}S$ is denoted the functor dual ([1]) to S.

Theorem 5. *Let $A \in \mathcal{BL}$, and let the sets in $\mathfrak{S}$ be relatively weakly bicompact. On the category $\mathcal{N}$ we have: a) $\mathcal{D}\Omega^{\mathfrak{S}}_A = \Omega_{A'_{\mathfrak{S}}}$; b) if A_τ is barrelled, then*

$$\mathcal{D}^2\Omega^{\mathfrak{S}}_A = \Omega_{A_\tau};$$

c) if A is reflexive, then Ω_A is reflexive.

Let us consider some consequences of these theorems.

Corollary 1. *Let X, Y be reflexive K-spaces, and let $\mathfrak{M}$ (co-

respectively $\mathfrak{N}$) is a topologizing system for X (respectively Y); the functors $\Phi^{\mathfrak{M}}_X$ and $\Phi^{\mathfrak{N}}_Y$ are defined on $\mathcal{DN}$.

Then: a) $\Phi_X^{\mathfrak{M}}$ is continuous on $\mathcal{DN}$ and regular on $\mathcal{FDM}$; b) if the systems $\mathfrak{M}$ and $\mathfrak{N}$ are bicompact, then $H_b(\Phi_X^{\mathfrak{M}}, \Phi_Y^{\mathfrak{N}})$ is isomorphic to $L_{\mathfrak{B}}(X_{\mathfrak{M}}, Y_{\mathfrak{N}})$; c) if $\mathfrak{M}$ is bicompact, then $\mathcal{D}\Phi_X^{\mathfrak{M}} = \Phi_X^{\mathfrak{B}}$; if $X_{\mathfrak{M}}$ is semireflexive, then $\mathcal{D}^2\Phi_X^{\mathfrak{M}} = \Phi_X^{\mathfrak{B}}$; if $X_{\mathfrak{M}}$ is a reflexive locally convex space, then $\Phi_X^{\mathfrak{M}}$ is reflexive.

Corollary 2. Let $A, B \in \mathcal{CL}$ (respectively $\mathcal{N}$); the functors $\Sigma_A, \Sigma_B, \Omega_A$ are defined on $\mathcal{N}$ (respectively $\mathcal{CL}$).

Then: a) Σ_A is continuous on $\mathcal{N}$ (respectively $\mathcal{CL}$) and regular on $\mathcal{RN}$ (respectively $\mathcal{LT}$); Ω_A is regular on $\mathcal{LT}$; b) $H_b(\Sigma_A, \Sigma_B)$ is isomorphic to $L_{\mathfrak{B}}(A, B)$; c) $\mathcal{D}\Omega_A = \Sigma_A$ on $\mathcal{N}$ (respectively $\mathcal{CL}$); $\mathcal{D}\Sigma_A = \Omega_A$ on $\mathcal{RN}$ (respectively $\mathcal{LT}$); d) Σ_A and Ω_A are reflexive on $\mathcal{RN}$ (respectively $\mathcal{LT}$); if A is reflexive, then Σ_A is reflexive on $\mathcal{N}$.

Using Theorem 2.5 of paper ([2]), we obtain

Corollary 3. For every functor defined on $\mathcal{N}$, there exists a continuous dual functor defined on the category of complete spaces satisfying the approximation condition.

Remark 1. Along with the definition of a dual functor given in (1), it is natural also to consider the following definition. Let S be a functor from $\mathcal{K}$ to $\mathcal{L}$; by ∂S denote the functor acting as follows: for $E \in \mathcal{L}$,

$$\partial SE = H_b(S, \Psi_E);$$

if $u \in L(E, F)$, then

$$((\partial Su)\alpha)_X = (\Psi_{Xu})\alpha_X,$$

where $\alpha \in H_b(S, \Psi_E)$,

$$\alpha_X \in L_{\mathfrak{B}}(SX, L_\varepsilon(E'_c, X)) \simeq L_{\mathfrak{B}}(SX, L_\varepsilon(X'_c, E')),$$

$$(\Psi_{Xu})\alpha_X \in L_{\mathfrak{B}}(SX, L_\varepsilon(X'_c, F') \simeq L_{\mathfrak{B}}(SX, L_\varepsilon(F'_c, X))) = L_{\mathfrak{B}}(SX, \Psi_{FX}).$$

With such a definition of duality, the notion of reflexivity of functors naturally arises, and it turns out that the functors Ω_A and Ψ_A are dual to one another and are reflexive on the category $\mathcal{LT}$ in this sense.

Remark 2. We indicate one more generalization of the results of ([1,2,5]). We shall call a functor t from a category $\mathcal{K} \subset \mathcal{L}$ to the category of partially ordered sets O topologizing if tE, for every E in $\mathcal{K}$, is a saturated system of absolutely convex bounded subsets of E, and if for $\alpha \in L(E, F)$ there is a canonical distribution α on tE. Examples of topologizing functors on $\mathcal{L}$: b, bE is the collection of all bounded subsets of E; p, pE is the collection of all precompact subsets of E; c, cE is the saturated system generated by the absolutely convex bicompact subsets of E; s, sE is the collection of finite-dimensional absolutely convex subsets of E. The pair $\langle \mathcal{K}, t \rangle$ will be called a topologized category. A functor $S : \mathcal{K}_1 \to \mathcal{K}_2$ will be called continuous from $\langle \mathcal{K}_1, t \rangle$ to $\langle \mathcal{K}_2, u \rangle$ if all the induced mappings $L_{tE}(E, F) \to L_{uSE}(SE, SF)$ are continuous. To functors in topologized categories one can extend the notions of regularity, duality, etc.

In conclusion I consider it my pleasant duty to express my deep gratitude to A. G. Pinsker and D. A. Raikov for their great attention to this work.

Mordovian State University

Received
18 I 1969

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Note: Figure translations are in progress. See original paper for figures.