

# SOLUTION OF THE MILNE PROBLEM FOR THE KINETIC EQUATION OF BHATNAGAR, GROSS, AND KROOK

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**Abstract**

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*MATHEMATICAL PHYSICS*

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## **SOLUTION OF THE MILNE PROBLEM FOR THE KINETIC EQUATION OF BHATNAGAR, GROSS, AND KROOK**

*(Presented by Academician G. I. Petrov on 15 VII 1968)*

In a number of cases it is necessary to solve the Milne problem for the simplest kinetic equation that takes into account the exchange of particle energies <sup>(1)</sup>

$$v \partial f / \partial x = (n f_0 - f) / \tau, \quad (1)$$

where  $f(v, x)$  is the distribution function;  $f_0$  is the Maxwell distribution;

$$n = \int_{-\infty}^{\infty} dv f(v, x);$$

$\tau$  is the constant relaxation time.

Consider, for example, a binary gas mixture, one component of which is absorbed by a wall. If the concentration of this component is small, then the distribution function of the second component may be regarded as Maxwellian ( $f_0$ ), with temperature  $T$  equal to the temperature of the wall. To determine the distribution function of the absorbing component it is necessary to solve the kinetic equation (1) with the boundary condition

$$f(v, 0) = 0 \quad \text{for } v > 0. \quad (2)$$

Problems of this kind may arise in chemical kinetics, as well as in the theory of low-temperature plasma and in neutron-transport theory. We note that equation (1) has been used, for example, to calculate the gas slip coefficient <sup>(2)</sup>.

Introducing the dimensionless variable  $\xi = x/v_0\tau$ ,  $u = -v/v_0$  ( $v_0 = \sqrt{2T/M}$ ) and performing a one-sided Fourier transform with respect to  $\xi$ , we obtain

$$F(u, k) = \frac{\frac{1}{\sqrt{\pi}} e^{-u^2} N(k) - \frac{u}{\sqrt{2\pi}} f(u, 0)}{1 + iku}, \quad (3)$$

where  $F(u, k)$  and  $N(k)$  are the Fourier transforms of the functions  $f(u, x) = v_0 f(v, x)$  and  $n(x)$ , respectively.

We shall seek a solution that grows algebraically as  $\xi \rightarrow \infty$  (3). In this case the functions  $F(u, k)$  and  $N(k)$  are analytic in the upper half-plane of the variable  $k$ . Then, for  $k = i/u$ ,  $u > 0$ , the equality holds

$$\frac{e^{-u^2}}{\sqrt{\pi}} N\left(\frac{i}{u}\right) = \frac{u}{\sqrt{2\pi}} f(u, 0). \quad (4)$$

Integrating (3) with respect to  $u$  over the interval  $(-\infty, \infty)$ , and using (2) and (4), we obtain

$$f(u)\lambda(u) = \frac{e^{-u^2}}{\sqrt{\pi}} \int_0^\infty \frac{dt t f(t)}{t - u}, \quad (5)$$

where

$$\lambda(u) = 1 + \frac{u}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{dt e^{-t^2}}{t - u} = 1 - \sqrt{\pi} u V(u), \quad f(u) \equiv f(u, 0). \quad (6)$$

The function  $V(u)$  is tabulated in (4).

The singular integral equation (5) is reduced by the usual methods (5) to the Riemann boundary-value problem

$$\Phi_+ = \frac{\Lambda_+}{\Lambda_-} \Phi_-, \quad (7)$$

where

$$\Phi(\eta) = \frac{1}{2} \int_0^\infty \frac{dt f(t)}{t - \eta}. \quad (8)$$

$$\Lambda(\eta) = 1 + \frac{\eta}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{dt e^{-t^2}}{t - \eta}. \quad (9)$$

Its solution is the function

$$\Phi(\eta) = C \frac{\exp \Gamma(\eta)}{\eta}, \quad (10)$$

where

$$\Gamma(\eta) = \frac{1}{2\pi i} \int_0^\infty \frac{\ln(\Lambda^+/\Lambda^-) - 2\pi i}{t - \eta} dt. \quad (11)$$

To determine the constant  $C$ , we expand  $\Phi(\eta)$ , defined in (8), in powers of  $1/\eta$ :

$$\Phi(\eta) = -\bar{u}/2\eta - \bar{u}^2/2\eta^2 - \bar{u}^3/2\eta^3, \quad (12)$$

where

$$\bar{u}^n = \int_0^\infty du u^n f(u).$$

Since the flux of particles to the wall is not equal to zero, the first term of this expansion is different from zero. Performing an analogous expansion in (10) and equating the corresponding terms of these expansions, we obtain

$$C = -\bar{u}/2, \quad (13)$$

$$\bar{u}^2 = l_0 \bar{u}, \quad (14)$$

$$\bar{u}^3 = (l_0^2/2 + l_1) \bar{u}, \quad (15)$$

where

$$l_k = \int_0^\infty \left( 1 - \frac{1}{\pi} \text{Arc tg} \frac{\sqrt{\pi} t e^{-t^2}}{\lambda(t)} \right) t^k dt. \quad (16)$$

Using (10) and (13), we write the solution of equation (5) in the form

$$f(u) = -\frac{\bar{u}}{\sqrt{\pi}} e^{-u^2} \frac{X_+(u)}{\Lambda_+(u)}, \quad (17)$$

where

$$X(\eta) = -\exp \Gamma(\eta)/\eta. \quad (18)$$

It can be shown that the function  $X(\eta)$  satisfies the relation

$$X(\eta)X(-\eta) = 2\Lambda(\eta). \quad (19)$$

Fig. 1: graph of  $f(u)$  versus  $u$

Figure 1: Fig. 1: graph of  $f(u)$  versus  $u$

Fig. 2

Figure 2: Fig. 2

It follows from this that

$$f(u) = \frac{2\bar{u}}{\sqrt{\pi}} \frac{e^{-u^2}}{X(-u)}, \quad (20)$$

Since the function  $X(-u)$  does not contain an integral in the sense of the principal value, it is easily found by numerical integration. The graph of the function  $f(u)$ , normalized to unity, is given in Fig. 1.

Fig. 1

Setting  $u = 0$  in (1), (19), and (20), we find the following relation between the particle density at the boundary  $n(0)$  and the flux into the wall  $I = \bar{u}v_0$ :

$$I = n(0)\sqrt{T/M} = \frac{1}{4}n(0)\bar{v}\sqrt{2\pi}. \quad (21)$$

The expression for the mean energy of a particle corresponding to the  $x$ -component of the velocity at the boundary  $x = 0$  can be obtained from (14):

$$\bar{E} = \frac{T}{n(0)} \overline{u^2} = \frac{l_0}{\sqrt{2}} T. \quad (22)$$

The mean energy carried by a particle into the electrode (for a Maxwellian distribution in the velocities  $v_y, v_z$ ) is equal to

$$1 + l_1 + \frac{l_0^2}{2} T.$$

The coefficients  $l_0, l_1$  were computed numerically and were found to be

$$l_0 = 1.016; \quad l_1 = 0.749.$$

**Fig. 2**

We now turn to determining the particle density near the wall. Substituting expressions (20), (21) into (4), we obtain

$$N\left(\frac{i}{u}\right) = \frac{n(0)}{\sqrt{\pi}} \frac{u}{X(-u)}. \quad (23)$$

Equality (23) can be analytically continued to the entire complex  $k$ -plane, since the function  $X(-u)$  is analytic everywhere except for the cut  $(-\infty, 0)$ .

$$N(k) = \frac{n(0)}{\sqrt{\pi}} \frac{i}{k} \frac{1}{X(-i/k)}. \quad (24)$$

Performing the inverse Fourier transform in (24), we obtain

$$n(\xi) = \sqrt{2} n(0) [\xi + l_0 - n_r(\xi)]. \quad (25)$$

The graph of the function

$$n_r(\xi) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{dt X(-t) \exp(-\xi/t - t^2)}{\Delta^+(t)\Delta^-(t)}$$

is shown in Fig. 2. As  $\xi \rightarrow \infty$ ,  $n_r(\xi) \rightarrow 0$ .

Taking into account that  $\xi = x/v_0\tau$ , we find that the extrapolated length in the case under consideration is equal to  $l_0 v_0 \tau$ .

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