FINITE CONTROL OVER DISTRIBUTED LINEAR SYSTEMS

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Abstract

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FINITE CONTROL OVER DISTRIBUTED LINEAR SYSTEMS

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In ([1]) the problem of finite control of systems with lumped parameters was considered. Here the problem of finite control for systems with distributed parameters is considered.

We shall explain the essence of the method for solving these problems by using as an example the solution of the simplest problem of finite control for a controllable wave system. The method described can be applied to other types of hyperbolic and parabolic equations. Let us consider a distributed controllable object that is described by a one-dimensional wave equation. Equations of this kind describe many technical systems (long electric transmission lines, surface waves in reservoirs, etc.).

Let the distribution of deviations of some physical controlled parameter Q be described by the function $Q(x, t)$, where x is the spatial coordinate and t the time coordinate. This function satisfies, inside the spatial interval $[0, S]$ and for $t > -\tau$ $(\tau \geq 0)$, the wave equation

$$\ddot{Q} = a^2 Q'', \qquad 0 < x < S, \qquad t > -\tau. \tag{1}$$

Here dots over Q denote derivatives with respect to t, and primes denote derivatives with respect to x.

For definiteness of the solution we prescribe the boundary conditions

$$Q(0, t) = u(t), \qquad Q(S, t) = 0, \qquad t \geq -\tau, \tag{2}$$

and the initial conditions

$$Q(x, -\tau) = Q_0(x), \qquad \dot{Q}(x, -\tau) = Q_1(x), \qquad 0 \leq x \leq S. \tag{3}$$

Here $u(t)$, $t \geq -\tau$, is the controlling action, or simply the control; $Q_0(x)$ and $Q_1(x)$ are prescribed (known) functions characterizing the initial distribution of deviations and of the rates of change of the distributed controlled physical parameter.

The problem formulated below will be called the problem of finite control. The problem is posed as follows.

Let a fixed time interval $[-\tau, \tau]$ be given, i.e., the quantity $\tau = T/2$ is regarded as prescribed, known, and fixed. On the interval $[-\tau, \tau]$ it is required to find a control $u(t)$, belonging to some class of controls K (this class will be discussed later), such that at time T the conditions

$$Q(x, \tau) = 0, \qquad \dot{Q}(x, \tau) = 0, \qquad 0 \le x \le S \tag{4}$$

are satisfied.

Condition (4) means that at time τ, and for $t > \tau$, the controlled system will be at rest (in a state of equilibrium). This problem may also be called the problem of bringing to rest, or damping oscillations. A control $u(t)$, $-\tau \le t \le \tau$, that solves this problem of finite control will be called a finite control. The term finite control

is explained by the fact that all processes are considered on the finite (finitary) time interval $[-\tau, \tau]$. The functions $Q(x, t)$ and $u(t)$, as functions of time outside the interval $[-\tau, \tau]$, shall, without loss of generality, be regarded as identically equal to zero (such functions in mathematics are called finitary).

The solution of the posed problem (as well as of other problems of finitary control) is based on a chain of operations including the Fourier transform in time of the system of equations (1), (2), the application of the Wiener–Paley theorem ([2],[3]), reduction to an interpolation problem for entire functions of a complex variable, the solution of this interpolation problem, and, finally, the application of the inverse Fourier transform. As a result, in many cases the answer, i.e. the desired finitary control $u(t)$, $-\tau \le t \le \tau$, can be obtained in closed analytic form as a concrete explicit function of time t.

For a clear description of this program it will be more convenient for us, without restricting generality, first to take $a = 1$, $S = \pi$.

Suppose now that the class K of finitary functions $u(t)$ on the interval $[-\tau, \tau]$ coincides with the class $L_2[-\tau, \tau]$. Therefore the functions $Q(x, t)$, for each fixed $x \in [0, \pi]$, also belong to the class $L_2[-\tau, \tau]$. After these remarks we proceed to the solution of the stated problem of finitary control.

We shall use the Fourier transform–direct and inverse–respectively in the form: direct

$$\tilde{y}(\omega) = F[y(t)] = \int_{-\infty}^{\infty} y(t) e^{-j\omega t}\, dt, \tag{5}$$

inverse

$$y(t) = F^{-1}[\tilde{y}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{y}(\omega) e^{j\omega t}\, d\omega. \tag{6}$$

Apply the Fourier transform to the system (1)–(3). We obtain

$$-\omega^2 \widetilde{Q}(x,\omega) - [j\omega Q_0(x) + Q_1(x)] e^{j\omega\tau} = \widetilde{Q}''(x,\omega),$$

$$0 \le x \le \pi, \qquad -\infty \le \omega \le \infty; \tag{7}$$

$$\widetilde{Q}(0,\omega) = \tilde{u}(\omega), \qquad \widetilde{Q}(\pi,\omega) = 0, \qquad -\infty \le \omega \le \infty. \tag{8}$$

As a result we obtain the ordinary differential equation (7) of second order with respect to the variable x for the function $\widetilde{Q}(x,\omega)$, for which it is necessary to solve the boundary-value problem (8), i.e. to find a particular solution of equation (7) satisfying conditions (8), regarding ω as a parameter ($-\infty \le \omega \le \infty$). Using the usual method of solving this problem, it is easy to find that the required function is

$$\widetilde{Q}(x,\omega) = \frac{\tilde{u}(\omega)\sin\omega(\pi - x) + b(\pi,\omega)\sin\omega x}{\sin\omega\pi} - b(x,\omega), \tag{9}$$

where

$$b(x,\omega) = \frac{1}{\omega}\int_0^x \sin\omega(x-y)[j\omega Q_0(y) + Q_1(y)]\, dy\, e^{j\omega\tau} =$$

$$= \left[j\int_0^x \sin\omega(x-y) Q_0(y)\, dy + \frac{1}{\omega}\int_0^x \sin\omega(x-y) Q_1(y)\, dy \right] e^{j\omega\tau}, \tag{10}$$

$$0 \le x \le \pi, \qquad -\infty \le \omega \le \infty.$$

Since by the condition of the finitary control problem the function $Q(x,t)$ must be a finitary function, i.e. $Q(x,t) = 0$ for $t \notin [-\tau,\tau]$,

$0 \le x \le \pi$ and $Q(x,t) \in L_2[-\tau,\tau]$ for all $x \in [0,\pi]$, then, by the Wiener–Paley theorem, the function $\widetilde{Q}(x,z)$, as a function of the complex argument $z = \omega + j\xi$, must be an entire function of fixed degree not exceeding T, independently of $x \in [0,\pi]$, and must be square-integrable on the real axis ω, $-\infty \le \omega \le \infty$, for all $x \in [0,\pi]$, i.e. $\widetilde{Q}(x,\omega) \in L_2(-\infty,\infty)$ for all $x \in [0,\pi]$.

By the same Wiener–Paley theorem, the converse assertion also turns out to be valid, and it is the basis for solving the problem of finite control. If the function $\widetilde{Q}(x,z)$ is an entire function of the complex argument $z=\omega+j\xi$ of fixed degree T, independent of $x\in[0,\pi]$, and if $\widetilde{Q}(x,\omega)\in L_2[-\infty,\infty]$ for all $x\in[0,\pi]$, then the inverse Fourier transform of this function $Q(x,t)=F^{-1}[\widetilde{Q}(x,\omega)]$ must be a finite function of t on the interval $[-T,T]$, i.e. $Q(x,t)=0$ for $t\in[-\tau,\tau]$, $x\in[0,\pi]$, and $Q(x,t)\in L_2[-\tau,\tau]$ for all $x\in[0,\pi]$.

Consequently, we must choose such a function $\tilde{u}(z)$ that it is an entire function of degree σ not exceeding T, $\tilde{u}(\omega)\in L_2[-\infty,\infty]$, and so that the entire right-hand side of formula (9) also represents an entire function of degree σ for all $x\in[0,\pi]$, and belongs to $L_2[-\infty,\infty]$ on the real axis ω also for all $x\in[0,\pi]$. But it is easy to see that $b(x,\omega)$ is an entire function of degree π, and it remains to require that the fraction in (9) also be an entire function. For this it is necessary to require that its numerator vanish at those points z_k at which the denominator vanishes, i.e. at the roots of the equation

$$\sin z\pi=0. \tag{11}$$

The roots of equation (11) are, obviously,

$$z_k=k,\qquad k=0,\ \pm1,\ \pm2,\ldots \tag{12}$$

It is obvious that for $z=z_0=0$ the fraction is already an entire function for any $x\in[0,\pi]$; therefore the condition must be satisfied

$$\tilde{u}(k)\sin k(\pi-x)+b(\pi,k)\sin kx=0,\qquad k=\pm1,\ \pm2,\ldots$$

The last equation can be transformed to the form

$$[\tilde{u}(k)-b(\pi,k)]\sin kx=0,\qquad k=\pm1,\ \pm2,\ldots,$$

whence, after simple transformations with account taken of (10), we obtain

$$\tilde{u}(k)=\beta_k,\qquad k=\pm1,\ \pm2,\ldots, \tag{13}$$

where

$$\beta_k=b(\pi,k)=\left[-j\int_0^\pi Q_0(x)\sin kx\,dx-\frac{1}{k}\int_0^\pi Q_1(x)\sin kx\,dx\right]e^{jk\tau}. \tag{14}$$

Let us note that the integrals in (14), up to the constant factor $2/\pi$, naturally independent of k, are the k-th Fourier coefficients in the expansions, respectively, of the functions $Q_0(x)$ and $Q_1(x)$ in a Fourier sine series on the interval $[0, \pi]$ for $k = 1, 2, \ldots$.

The conditions (13) express an interpolation problem for the desired entire function $\tilde{u}(z)$. Thus, the problem has been reduced to determining an entire function $\tilde{u}(z)$ of degree σ not exceeding T, which solves the interpolation problem (13) and $\tilde{u}(\omega) \in L_2(-\infty, \infty)$.

It may already be said that the smallest possible degree σ of the desired entire function $\tilde{u}(z)$ is equal to the degree of the function $\sin \pi z$, i.e. $\sigma = \pi$. Consequently, the minimal interval of time t outside which the inverse transform of the function $\tilde{u}(\omega)$ is identically equal to zero is the interval $[-\pi, \pi]$, i.e. $\tau = \pi$. If the initial conditions (3) are prescribed at $t = 0$

(and not at $t = \tau$), then the desired finite control $u(t)$ must be equal to zero for $t < 0$, and the obtained inverse Fourier transform must be shifted in time to the right by the amount $\tau = \pi$, and, consequently, the minimal settling time of the system is equal to $T = 2\tau = 2\pi$—the time for a wave to traverse twice the length of the distributed system.

The function $\tilde{v}(z)$ solving the interpolation problem (13) can be found by the Lagrange formula ([4,5])

$$\tilde{v}(z) = \sum_{k=-\infty}^{\infty}{}' \frac{\beta_k \sin \pi z}{(-1)^k \pi (z-k)} + \varphi(z) = \sum_{k=-\infty}^{\infty}{}' \beta_k \frac{\sin \pi (z-k)}{\pi (z-k)} + \varphi(z), \qquad (15)$$

where the prime on the summation sign denotes the absence of the term corresponding to $k = 0$, and $\varphi(z)$ is an arbitrary entire function of degree not higher than π and such that $\varphi(\omega) \in L_2(-\infty, \infty)$ and $\varphi(k) = 0,\ k = \pm 1, \pm 2, \ldots$.

Formula (15), thus, describes the entire set of controls solving the finite-control problem. Any desired finite control can be obtained from it by a suitable choice of the function $\varphi(z)$.

This arbitrariness in the choice of $\varphi(z)$ can be used to solve various optimal-control problems under the condition of settling of the oscillating system. Let us note without proof that if $\varphi(z) = 0$, then the finite control obtained is optimal in the system in the sense of the minimum of the integral

$$\int_0^{2\pi} u^2(t)\, dt,$$

which can be interpreted as the energy of the controlling action. Let us find the finite control in the case $\varphi(z) = 0$. It is easy to see that in this case, taking (15) into account, we obtain

$$u(t) = \operatorname{Re} v(t - \pi) = \operatorname{Re} F^{-1} \left[\tilde{v}(\omega) e^{-j\pi\omega}\right] =$$

$$= \frac{1}{2}\bar{Q}_0(t) + \frac{1}{2}\int_0^t \bar{Q}_1(y)\,dy - \frac{1}{2\pi}\int_0^\pi (\pi - t)Q_1(y)\,dy, \qquad 0 \le t \le 2\pi, \qquad (16)$$

where

$$\bar{Q}_0(t) = \begin{cases} Q_0(t), & 0 \le t \le \pi, \\ -Q_0(2\pi - t), & \pi < t \le 2\pi; \end{cases} \qquad \bar{Q}_1(t) = \begin{cases} Q_1(t), & 0 \le t \le \pi, \\ -Q_1(2\pi - t), & \pi < t \le 2\pi. \end{cases}$$

The described method for solving the simplest problem of settling spatially distributed oscillations can readily be extended to more general cases, when there is a disturbing force $f(x, t)$ in the right-hand side of equation (1) and more general boundary conditions in the form of linear combinations of displacement and velocity, when there are several controlling actions, and, finally, this method can be extended to multidimensional equations ([6]).

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Note: Figure translations are in progress. See original paper for figures.