ON THE ORDER OF GROWTH OF SUMS OF INDEPENDENT RANDOM VARIABLES

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Abstract

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MATHEMATICS

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ON THE ORDER OF GROWTH OF SUMS OF INDEPENDENT RANDOM VARIABLES

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Let $\{X_n\}$ $(n = 1, 2, \ldots)$ be a sequence of independent random variables,

$$S_n = \sum_{k=1}^{n} X_k.$$

The set of functions $\psi(x)$ such that each $\psi(x)$ is positive and nondecreasing in the region $x > x_0$ for some x_0, and the series $\sum \frac{1}{n\psi(n)}$ converges (diverges), will be denoted by Ψ_c (respectively Ψ_d)*.

1. Theorem 1. Let $g(x)$ be an even continuous function, positive and strictly increasing in the region $x > 0$, with $g(x) \to \infty$ as $x \to \infty$. Suppose that one of the following two conditions is satisfied:

(A) $x/g(x)$ is nondecreasing in the region $x > 0$.

(B) $x/g(x)$ and $g(x)/x^2$ are nonincreasing in the region $x > 0$.

Suppose, further,**

$$Eg(X_n) < \infty \qquad (n = 1, 2, \ldots), \tag{1}$$

$$A_n \to \infty, \tag{2}$$

where

$$A_n = \sum_{k=1}^{n} Eg(X_k). \tag{3}$$

In addition, in the case when condition (B) is satisfied, assume that

$$EX_n = 0, \qquad (n = 1, 2, \ldots). \tag{4}$$

Then

$$S_n = o(g^{-1}(A_n \psi(A_n))) \quad \text{a.s.} \tag{5}$$

for any function $\psi(x) \in \Psi_c$. Here g^{-1} denotes the function inverse to g.

The following theorem shows that if, instead of $\psi(x) \in \Psi_c$, one takes a more slowly increasing function $\psi(x) \in \Psi_d$, then relation (5) may fail.

Theorem 2. Let $g(x)$ be a function satisfying the conditions of Theorem 1. For any function $\psi(x) \in \Psi_d$ there exists a sequence of independent random variables $\{X_n\}$ satisfying conditions (1), (2), (4) and the condition

$$\limsup |S_n| / g^{-1}(A_n \psi(A_n)) > 0 \quad \text{a.s.} \tag{6}$$

2. We now give simply formulated consequences of these theorems, corresponding to the case $g(x) = |x|^p, \ 0 < p \leq 2$.

Theorem 3. Let

$$E|X_n|^p < \infty \qquad (n = 1, 2, \ldots) \tag{7}$$

for some positive $p \leqslant 2$. Let

$$A_n = \sum_{k=1}^{n} E|X_k|^p \to \infty. \tag{8}$$

Then in the case $0 < p < 1$ we have

$$S_n = o\left([A_n \psi(A_n)]^{1/p}\right) \quad \text{a.s.} \tag{9}$$

* Here and everywhere below, $\sum f(n)$ denotes summation over all positive integers n

for which the values $f(n)$ are defined and nonnegative. In the definitions of the sets Ψ_c and Ψ_d, the value x_0 is not assumed to be the same for different functions ψ.

** Everywhere the indicated limits are as $n \to \infty$, unless otherwise stated. The notation a.s. means almost s for any function $\psi(x) \in \Psi_c$, while in the case $1 \leqslant p \leqslant 2$ the same assertion is true under the additional condition (4).

Theorem 4. For any function $\psi(x) \in \Psi_d$ and any positive number $p \leqslant 2$, there exists a sequence of independent random variables $\{X_n\}$ satisfying conditions (4), (7), (8) and the condition

$$\limsup |S_n|/[A_n\psi(A_n)]^{1/p} > 0 \quad \text{a.s.} \tag{10}$$

3. The case $p = 2$ is of special interest. In this case, Theorems 3 and 4 immediately imply the following results.

Theorem 5. Let $\{X_n\}$ be a sequence of independent random variables with finite variances $DX_n = E(X_n^2) - (EX_n)^2$ $(n = 1, 2, \dots)$. Put

$$B_n = \sum_{k=1}^{n} DX_k.$$

If $B_n \to \infty$, then

$$S_n - ES_n = o\left(\sqrt{B_n\psi(B_n)}\right) \quad \text{a.s.} \tag{11}$$

for any function $\psi(x) \in \Psi_c$.

Theorem 6. For any function $\psi(x) \in \Psi_d$ there exists a sequence of independent random variables $\{X_n\}$ with mathematical expectations equal to zero and finite variances such that

$$B_n = \sum_{k=1}^{n} DX_k \to \infty$$

and

$$\limsup |S_n|/\sqrt{B_n\psi(B_n)} > 0 \quad \text{a.s.} \tag{12}$$

It follows from Theorem 5 that, for sums S_n of independent random variables with finite variances and unboundedly increasing variance of the sum $B_n = DS_n$, the following estimates of the order of growth hold, each of which is stronger than the preceding one: for any $\varepsilon > 0$,

$$S_n - ES_n = o\left(B_n^{1/2+\varepsilon}\right) \quad \text{a.s.},$$

$$S_n - ES_n = o\left(B_n^{1/2}(\log B_n)^{1/2+\varepsilon}\right) \quad \text{a.s.},$$

$$S_n - ES_n = o\left(B_n^{1/2}(\log B_n)^{1/2}(\log\log B_n)^{1/2+\varepsilon}\right) \quad \text{a.s.},$$

and so on. By virtue of Theorem 6, in these estimates one cannot replace $\varepsilon > 0$ by zero without introducing additional assumptions.

4. The results obtained can be used in studying conditions for the applicability of the strong law of large numbers with the simplest normalization, namely in the form

$$\frac{1}{n}(S_n - ES_n) \to 0 \quad \text{a.s.} \tag{13}$$

As is known, the condition

$$B_n = o(n^2) \tag{14}$$

is sufficient for the applicability of the weak law of large numbers, i.e. for the convergence of $\frac{1}{n}(S_n - ES_n)$ to zero in probability. With the help of Theorem 5 one can indicate a strengthening of condition (14) that ensures the applicability of the strong law of large numbers.

Theorem 7. If

$$B_n = O(n^2/\psi(n)) \tag{15}$$

for some function $\psi(x) \in \Psi_c$, then relation (13) holds.

On the other hand, whatever the function $\psi(x) \in \Psi_d$ for which $n/\psi(n)$ does not decrease in the domain $n > n_0$, for some n_0, there exists a sequence of independent random variables $\{X_n\}$ with finite variances for which (15) is satisfied, but

$$P\left\{\frac{1}{n}(S_n - ES_n) \to 0\right\} = 0.$$

Let us give one consequence of Theorem 7. The condition $B_n = O(n^2/(\log n)^{1+\delta})$ for some $\delta > 0$ is sufficient for the applicability of the strengthened law of large numbers to a sequence of independent random variables $\{X_n\}$, i.e., for relation (13). If, however, the condition $B_n = O(n^2/\log n)$ or even the stronger condition $B_n = O(n^2/\log n \log\log n)$ is satisfied, then relation (13) may fail to hold.

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