DIFFRACTION ON A BOUNDED BODY

MATHEMATICAL PHYSICS

1969

SovietRxiv

Abstract

Full Text

UDC 517.496:534.232.26

MATHEMATICAL PHYSICS

A. G. SVESHNIKOV

DIFFRACTION ON A BOUNDED BODY

(Presented by Academician A. N. Tikhonov, 12 V 1968)

Of great practical interest is the solution of problems of the mathematical theory of diffraction in those cases when the length of the incident wave is commensurable with the characteristic dimensions of the diffraction object. In these cases asymptotic methods of investigation, based on the geometrical-optics representation, prove inapplicable, and until recently the most effective results have been obtained by means of the method of integral or functional equations ([1]–[3]). However, the application of these methods in cases where the properties of the exterior medium are arbitrary functions of the coordinates is associated with considerable difficulties.

In the present paper an algorithm is proposed for the numerical solution of a rather general class of diffraction problems, reducible to the solution of a boundary-value problem for a system of ordinary differential equations. In its conceptual aspect the proposed method is closely connected with the method we developed for the investigation of oscillations in irregular waveguides ([4]). For simplicity of exposition we shall restrict ourselves to the scalar case.

1. Consider the problem of diffraction by a body T, bounded by a smooth closed surface S, of a scalar field produced by sources distributed over this surface. We shall assume that impedance boundary conditions are satisfied on S. Let the characteristics of the material properties of the medium outside S be arbitrary continuously differentiable functions of the coordinates, and let us suppose that inside the body T one can specify such a point O that outside the sphere Σ_{R_0} of radius R_0 with center at this point the material characteristics of the medium are constant. The mathematical formulation of the problem under consideration reduces to determining, outside the body T, a solution of the equation

$$L[u] \equiv \operatorname{div}\{p(x,y,z)\operatorname{grad} u\} + q(x,y,z)u = 0, \tag{1}$$

which outside the sphere Σ_{R_0} becomes the equation

$$\Delta u + k_0^2 u = 0, \tag{1'}$$

where k_0 = const. On the surface S the function u satisfies the boundary condition

$$\partial u/\partial n - \alpha(P)u\big|_{P\in S} = \Phi(P), \tag{2}$$

where $\alpha(P)$ and $\Phi(P)$ are prescribed functions, and at infinity it satisfies radiation conditions, which may be written in the form of the Sommerfeld conditions ([5]) or in any other equivalent form corresponding to the condition of absence of waves arriving from infinity. We shall assume that the coefficients in (1) and (2) and the surface S satisfy sufficient smoothness conditions for the existence of a classical solution of the posed problem ([6]).

By virtue of the radiation conditions, the function u outside Σ_{R_0} in a spherical coordinate system with origin at the point O can be written in the form (the time depen-

dependence $e^{-i\omega t}$)

$$u\big|_{r\geq R_0} = \sum_{n,m} T_{n,m}\, \zeta_n^{(1)}(k_0 r) Y_n^{(m)}(\theta,\varphi), \tag{3}$$

where $T_{n,m}$ are constant coefficients determining the amplitudes of the spherical waves diverging from the body T, and $\{Y_n^{(m)}\}$ is a system of spherical functions orthonormal on the unit sphere.

Let us note that in many physical problems the principal interest is the determination of the coefficients $T_{n,m}$. The present work is devoted to this question.

In view of (3), on Σ_{R_0} the relation holds

$$\operatorname{Im} \iint_{\Sigma_{R_0}} \frac{\partial u}{\partial n} u^* \, d\sigma = \frac{1}{k_0} \sum_{n,m} |T_{n,m}|^2. \tag{4}$$

Let us note that, by Green' s formula, relation (4) is also preserved on any smooth closed surface S_0 containing the sphere Σ_{R_0} inside it.

Consider, in the region bounded by the surfaces Σ_{R_0} and S_0, the system of functions $v_k = \zeta_n^{(1)}(k_0 r) Y_n^{(m)}(\theta,\varphi)$. On S_0 these functions form a complete system (7). The relation holds

$$\iint_{S_0} \frac{\partial u}{\partial n} v_l^* \, d\sigma = \sum_{k=1}^{\infty} \gamma_{l,k} T_k \quad (l = 1, 2, ...), \tag{5}$$

where T_k are the coefficients of the expansion of the function u on S_0 in the system $\{v_k\}$, and the coefficients

$$\gamma_{l,k} = \iint_{S_0} \frac{\partial v_k}{\partial n} v_l^* \, d\sigma$$

are connected by the conditions $\gamma_{k,l} = \gamma_{l,k}^*$ for $k \neq l$, $\operatorname{Im} \gamma_{k,k} = 1/k_0$, which also ensures the fulfillment of (4) on S_0. Relations (5) have the meaning of partial radiation conditions, which the solution of the problem under consideration must satisfy on the surface S_0.

Thus, the problem consists in constructing, in the region D bounded by the surfaces S and S_0, a solution of equation (1), (1′), satisfying conditions (2) and (5).

2. To construct an approximate solution of the stated problem, following (4), we map the region D onto the spherical layer K, $\{1 \leq \xi \leq 2,\ 0 \leq \theta' \leq \pi,\ 0 \leq \varphi' \leq 2\pi\}$, so that the surface S goes over into the sphere $\xi = 1$, and S_0 into $\xi = 2$.

We shall seek the approximate solution in the form

$$u_N(\xi, \theta', \varphi') = \sum_{n=1}^{N} A_n(\xi) \chi_n(\theta', \varphi'), \tag{6}$$

where the functions $\chi_n(\theta', \varphi')$ form a complete and orthonormal system on the surface $\xi = \text{const}$.

Let us note that both the mapping of D onto K and the choice of the system $\{\chi_n\}$ can be carried out in many ways. Between the functions $V_k(2, \theta', \varphi')$, which under the given mapping pass into the functions $v_k = \zeta_n^{(1)}(k_0 r) Y_n^{(m)}(\theta, \varphi)\big|_{S_0}$, and the functions χ_n there are established relations

$$\chi_n(\theta', \varphi') = \sum_k a_{n,k} V_k(2, \theta', \varphi'), \tag{7}$$

where the coefficients $a_{n,k}$ are determined by standard methods.

Following the main idea of the Galerkin method, we shall determine the functions $A_n(\xi)$ from the solution of a boundary-value problem for a system of ordinary differential equations, requiring the fulfillment of the relations

$$\iint_{\xi=\text{const}} L[u_N] \chi_n^* \, d\sigma' = 0 \quad (1 < \xi < 2,\ n = 1, 2, \dots, N); \tag{8}$$

$$\iint_{\xi=1} \left\{ \frac{\partial u_N}{\partial n} - \alpha u_N - \Phi \right\}_S \chi_n^* \, d\sigma' = 0 \qquad (n = 1, 2, \dots, N); \tag{9}$$

$$\iint_{\xi=2} \frac{\partial u_N}{\partial n}\bigg|_{S_0} \chi_n^* \, d\sigma' = \sum_k b_{n,k} T_k^{(N)} \qquad (n = 1, 2, \dots, N), \tag{10}$$

where $b_{n,k} = \sum_m a_{n,m}^* \gamma_{m,k}$, and $T_k^{(N)}$ are the coefficients of the expansion of the function u_N on the surface S_0 in the functions v_k

$$u_N\big|_{S_0} = \sum_{k=1}^{\infty} T_k^{(N)} v_k. \tag{11}$$

As is easily seen, by virtue of (7) these coefficients satisfy the relations

$$T_k^{(N)} = \sum_{n=1}^{N} a_{n,k} A_n(2). \tag{12}$$

Therefore (10) takes the form

$$\iint_{\xi=2} \frac{\partial u_N}{\partial n}\bigg|_{S_0} \chi_n^* \, d\sigma' = \sum_{m=1}^{N} c_{n,m} A_m(2) \qquad (n = 1, 2, \dots, N), \tag{13}$$

where

$$c_{n,m} = \sum_{k,l} \gamma_{k,l} a_{n,k}^* a_{m,l}. \tag{14}$$

The resulting boundary-value problem (8), (9), (13) can be solved numerically by standard methods; from it the coefficients $T_k^{(N)}$ of the expansion of the approximate solution in spherical waves are also determined.

3. The convergence of the approximate solution u_N to the exact one as $N \to \infty$ is established by the methods developed earlier ([8]), proceeding from the energy relation for the approximate solution, which is a consequence of (8)–(10), and from the properties of the system of functions $\{\chi_n\}$. In this case the coefficients $T_k^{(N)}$, as $N \to \infty$, converge to the coefficients T_k of the expansion of the exact solution in spherical waves.

4. The proposed algorithm allows arbitrariness not only in the choice of the mapping D onto K and of the functions $\{\chi_n\}$, but also in the choice of the surface S_0 on which the partial radiation conditions (5) are prescribed. All these questions are determined by the requirement that the algorithm be simple to implement in solving particular problems. In a number of cases it proves convenient to choose the sphere Σ_{R_0} as the surface S_0, and to construct the functions $\{\chi_n\}$ by orthogonalizing the system $\{V_k(2, \theta', \varphi')\}$ on the surface $\xi = 2$.

5. The considerations carried out are readily extended to the case of an inhomogeneous equation (1) and to vector diffraction problems.

Moscow State University
named after M. V. Lomonosov

Received
17 IV 1968

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Note: Figure translations are in progress. See original paper for figures.