

# LIE SUBGROUPS OF THE LORENTZ GROUP WITH A FINITE GROUP OF COMPONENTS

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**Abstract**

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**MATHEMATICS**

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**LIE SUBGROUPS OF THE LORENTZ GROUP WITH A FINITE GROUP OF COMPONENTS**

*(Presented by Academician L. I. Sedov on 18 X 1968)*

Consider the Lorentz group  $G$  of linear transformations of the variables  $x^i$  ( $i = 1, 2, 3, 4$ ) that leave invariant the quadratic form  $x^{12} + x^{22} + x^{32} - x^{42}$ . The infinitesimal operators of its connected component of the identity, called the proper Lorentz group  $H_0$ , have the form

$$\begin{aligned} X_1 &= x^2\partial_3 - x^3\partial_2, & X_2 &= x^3\partial_1 - x^1\partial_3, & X_3 &= x^1\partial_2 - x^2\partial_1, \\ X_4 &= x^1\partial_4 + x^4\partial_1, & X_5 &= x^2\partial_4 + x^4\partial_2, & X_6 &= x^3\partial_4 + x^4\partial_3, \end{aligned}$$

where  $\partial_i = \partial/\partial x^i$ .

To study subgroups of the group  $G$ , consider its spinor representation by the group  $A$  of matrices of the form

$$\begin{vmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{vmatrix}, \quad \begin{vmatrix} 0 & \alpha \\ \bar{\alpha} & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & \alpha \\ -\bar{\alpha} & 0 \end{vmatrix}, \quad \begin{vmatrix} \alpha & 0 \\ 0 & -\bar{\alpha} \end{vmatrix},$$

where the group  $H_0$  is put into correspondence with the group of complex unimodular second-order matrices  $\alpha$  according to the law

$$c' = \alpha c \alpha^*, \quad c = \begin{vmatrix} x^4 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^4 - x^3 \end{vmatrix}.$$

The bar denotes complex conjugation, and the asterisk Hermitian conjugation. In this representation, to the reflection of  $x^\alpha$  ( $\alpha = 1, 2, 3$ ) there correspond the matrices

$$\pm \begin{vmatrix} 0 & \tau \\ \tau & 0 \end{vmatrix}, \quad \text{where } \tau = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix},$$

and to the reflection of  $x^4$ , the matrices

$$\pm \begin{vmatrix} 0 & \tau \\ -\tau & 0 \end{vmatrix}.$$

Since this representation is two-valued, we shall consider only subgroups of the group  $A$  containing the matrices  $\pm e$  ( $e$  is the identity matrix), corresponding to subgroups of the group  $G$ .

All finite subgroups of the group  $G$  are conjugate in it to finite subgroups of its orthogonal subgroup containing orthogonal transformations of the variables  $x^\alpha$  and the reflection of  $x^4$ , which are known <sup>(1, 2)</sup>, since a transformation  $v$ , under which the matrices  $a_k$  of the elements  $m_k$  ( $k = 1, \dots, r$ ) of a finite subgroup are reduced to unitary form corresponding to an orthogonal subgroup of the group  $G$ , determined by the relation

$$v^*v = \frac{1}{r} \sum_{k=1}^r m_k^* m_k$$

<sup>(3)</sup>, is reduced to a transformation belonging to the identity component of the group  $A$ , i.e., corresponding to a proper Lorentz transformation from  $H_0$ .

The classification of real Lie subalgebras of the Lie algebra of the Lorentz group is known <sup>(4, 5)</sup>. From it it is not difficult to reconstruct, up to conjugacy in the group  $G$ , all connected subgroups  $H_\mu$  ( $\mu = 0, 1, \dots, 13$ ). To find Lie subgroups of the group  $G$  of positive dimension with a finite group of components, since the identity component of a Lie group is its invariant subgroup <sup>(6)</sup>, we consider, within the spinor representation, the normalizers  $N_\mu$ ,  $bab^{-1} = a'$ ,  $a, a' \in H_\mu$ ,  $b \in N_\mu$ , of these connected subgroups in the group  $G$ , which itself is a normalizer for  $H_0$ .

In this case, for the groups  $H_\mu$ , the matrices  $\alpha$  of which have the form  $\begin{vmatrix} x & y \\ 0 & 1/x \end{vmatrix}$ ,  $x \neq 0$ , the following cases are possible:

1.  $y$  is an arbitrary complex number; in this case the matrices  $\alpha$  corresponding to the normalizer have the form  $\begin{vmatrix} p & q \\ 0 & 1/p \end{vmatrix}$ ,  $p \neq 0$ , where  $p$  and  $q$  are arbitrary complex numbers;
2.  $\text{Im } y = 0$ ; for  $\text{Im } x = 0$  we obtain, for the normalizer, matrices  $\alpha$  of the form  $\begin{vmatrix} i\rho & i\sigma \\ 0 & -i/\rho \end{vmatrix}$  or  $\begin{vmatrix} \rho & \sigma \\ 0 & 1/\rho \end{vmatrix}$ , where  $\text{Im } \rho = \text{Im } \sigma = 0$ ; for  $x = \pm 1$ , of the form  $\begin{vmatrix} i\rho & q \\ 0 & -i/\rho \end{vmatrix}$  or  $\begin{vmatrix} \rho & q \\ 0 & 1/\rho \end{vmatrix}$ , where  $\text{Im } \rho = 0$ ;
3.  $y = 0$  gives the same matrices of the form  $\begin{vmatrix} p & 0 \\ 0 & 1/p \end{vmatrix}$  or  $\begin{vmatrix} 0 & q \\ -1/q & 0 \end{vmatrix}$ .

The first case corresponds to the groups  $H_1, H_2(|x| = 1), H_3(\text{Im } x = 0), H_4(x = \pm 1)$ , and  $H_5(x = \pm e^{(k+i)\varphi}, \text{Im } \varphi = 0, k > 0)$ ; the second to  $H_6(\text{Im } x = 0)$  and  $H_7(x = \pm 1)$ ; the third to  $H_8, H_9(|x| = 1), H_{10}(x = \pm e^{(k+i)\varphi}, \text{Im } \varphi = 0, k > 0)$ , and  $H_{11}(\text{Im } x = 0)$ .

The groups  $H_{12} \left( \alpha = \begin{vmatrix} x & y \\ -\bar{y} & \bar{x} \end{vmatrix} \right)$  and  $H_{13} \left( \alpha = \begin{vmatrix} x & y \\ \bar{y} & \bar{x} \end{vmatrix} \right)$  give normalizers whose matrices  $\alpha$  have, respectively, the form  $\begin{vmatrix} x & y \\ -\bar{y} & \bar{x} \end{vmatrix}$  and  $\begin{vmatrix} x & y \\ \bar{y} & \bar{x} \end{vmatrix}$ , or  $\begin{vmatrix} x & y \\ -\bar{y} & -\bar{x} \end{vmatrix}$ .

If the matrix  $\bar{a}$  does not belong to  $H_\mu$  simultaneously with  $a$ , then the matrices of the normalizer  $N_\mu$  have the form  $\begin{vmatrix} a & 0 \\ 0 & \pm \bar{a} \end{vmatrix}$ ; if  $\bar{a}$  belongs to  $H_\mu$ , then they have the form  $\begin{vmatrix} a & 0 \\ 0 & \pm \bar{a} \end{vmatrix}$  or  $\begin{vmatrix} 0 & a \\ \pm \bar{a} & 0 \end{vmatrix}$ . For the groups  $H_5$  and  $H_{10}$ ,  $\bar{a}$  does not belong to the group.

All the normalizers  $N_\mu$  found, for each of their subgroups with a finite group of components, make it possible to choose representatives of conjugacy classes modulo  $H_\mu$  so that they form a finite group isomorphic to the quotient group modulo  $H_\mu$ . Therefore, for the classification up to conjugacy of subgroups of the group  $G$  with a finite group of components, it suffices, for a fixed group  $H_\mu$ , to list all finite subgroups of  $N_\mu$  not conjugate in it, whose intersection with  $H_\mu$  is equal to  $\pm e$ , the element corresponding to the identity of the group  $G$ . All of them will give distinct non-conjugate subgroups of  $G$  with a finite group of components.

Taking into account the results obtained and the explicit form of the matrices of the finite groups in the spinor representation, we shall write out all possible finite groups isomorphic to the component groups of subgroups of the group  $G$ . In parentheses are indicated the infinitesimal operators, and the notation of finite groups according to Shubnikov [1]\* is used. For  $H_0, H_1(X_3, X_6, X_4 - X_2, X_1 + X_5), H_2(X_3, X_4 - X_2, X_1 + X_5), H_{12}(X_1, X_2, X_3)$  we have  $1, 1 : 2, \bar{2}, 1 \cdot m, \bar{2} \cdot m$ ; for  $H_3(X_6, X_4 - X_2, X_1 + X_5), H_4(X_4 - X_2, X_1 + X_5)$   $n, n : 2, \bar{2}n \cdot m, n \cdot m, m \cdot n : m, 2n \cdot m (n = 1, 2, \dots)$ ;

for  $H_5(X_3 + kX_6, X_4 - X_2, X_1 + X_5)$   $n, \bar{2}n, n : m (n = 1, 2, \dots, k > 0)$

for  $H_6(X_6, X_4 - X_2), H_7(X_4 - X_2)$   $1, 2, 1 : 2, \bar{2}, 2 : 2, 1 : m, 2 : m, 1 \cdot m, 2 \cdot m, m \cdot 1 : m, m \cdot 2 : m, \bar{2} \cdot m, 1 : 2', 1 \cdot m', m' \cdot 1 : m, \bar{2} \cdot m'$ ;

for  $H_8(X_3, X_6), H_9(X_3), H_{13}(X_3, X_4, X_5)$   $1, \bar{1}, 1 : 2, 1 : \bar{2}, 1 : 2, \bar{2}, \bar{2} \times \bar{1}, 1 \cdot m, 1 \cdot \underline{m}, \bar{1} \cdot m, \bar{2} \cdot m, 2 \cdot \underline{m}, \bar{2} \cdot m, \bar{2} \cdot \underline{m}, \bar{2} \cdot m \times \bar{1}$ ;

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\*  $n$  denotes a rotation about the axis  $x^3$  through the angle  $2\pi/n$ ,  $1 : 2$  —about the axis  $x^2$  through  $\pi$ ,  $: m$  and an overbar denote reflection of  $x^3$ ,  $\cdot m - x^2$ , an

underbar  $-x^4$ ,  $1 : 2'$  –rotation about the axis  $x^1$  through the angle  $\pi$ , and  $\cdot m'$  –reflection of  $x^1$ .

for  $H_{10}(X_3 + kX_6)$   $n$ ,  $n : 2$ ,  $\overline{2n}$ ,  $n : m$ ,  $n \cdot m$ ,  $m \cdot n : m$ ,  $\overline{2n} \cdot m$  ( $n = 1, 2, \dots$ ,  $k > 0$ );

for  $H_{11}(X_6)$   $n$ ,  $2n$ ,  $n \times 1$ ,  $n : 2$ ,  $n : \underline{2}$ ,  $\underline{2n} : 2$ ,  $n : 2 \times \underline{1}$ ,  $\overline{2n}$ ,  $\underline{2n}$ ,  $\overline{2n} \times 1$ ,  $n : m$ ,  $n : \underline{m}$ ,  $\underline{2n} : m$ ,  $2n : \underline{m}$ ;  $n : m \times \underline{1}$ ,  $n \cdot m$ ,  $n \cdot \underline{m}$ ,  $\underline{2n} \cdot m$ ,  $n \cdot m \times \underline{1}$ ,  $m \cdot n : m$ ,  $\underline{m} \cdot n : m$ ,  $\underline{m} \cdot n : \underline{m}$ ,  $m \cdot n : \underline{m}$ ,  $m \cdot \underline{2n} : m$ ,  $m \cdot \underline{2n} : \underline{m}$ ,  $m \cdot n : m \times 1$ ,  $\overline{2n} \cdot m$ ,  $\overline{2n} \cdot \underline{m}$ ,  $\underline{2n} \cdot m$ ,  $\underline{2n} \cdot m$ ,  $\overline{2n} \cdot m \times 1$  ( $n = 1, 2, \dots$ ).

Thus, taking into account the finite subgroups of the group  $G$ , we obtain altogether 207 distinct types of subgroups with finite component group, 86 of which depend on the natural number  $n$ . In particular, for  $\mu = 9, 12$  one obtains the limiting groups known in crystallography  $(\underline{1}, \underline{2})$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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