LONG UNSTEADY WAVES IN A ROTATING BASIN

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Abstract

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LONG UNSTEADY WAVES IN A ROTATING BASIN

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We investigate the influence of the Coriolis force on long unsteady waves in a plane axisymmetric hydrodynamic model. Suppose that a plane unbounded basin of depth h, filled with an ideal incompressible fluid, rotates with constant angular velocity ω about a vertical axis. The z-axis of the cylindrical coordinate system coincides with the axis of rotation of the basin. We write the linear equations of the theory of long waves for axisymmetric motions in the form ([1])

$$\begin{gathered}\frac{\partial V_r}{\partial t} - 2\omega V_s + g\frac{\partial \zeta}{\partial r} = 0,\\ \frac{\partial V_s}{\partial t} + 2\omega V_r = 0,\\ \frac{\partial \zeta}{\partial t} + \frac{h}{r}\frac{\partial}{\partial r}(rV_r) = 0.\end{gathered} \tag{1}$$

Here V_r and V_s are the radial and tangential components of velocity; ζ is the elevation of the free surface, measured from the position of relative equilibrium; r is the distance from the origin to the point of observation; t is time.

Let, up to the instant $t = 0$, the fluid be at rest; then at the point $r = 0$ a source is switched on with discharge $Qf(t)$, where $f(t)$ is an arbitrary dimensionless function of time. Then the solution of the system of partial differential equations (1) must satisfy the initial conditions

$$V_r(r,0) = V_s(r,0) = \zeta(r,0) = 0 \tag{2}$$

and the limiting relation

$$\lim_{r\to 0}(2\pi r h V_r) = Qf(t). \tag{3}$$

To solve problem (1)–(3), we carry out the Laplace-Carson transform in time. Solving the equations for the transforms, we find

$$\begin{aligned}\zeta(r,p) &= \frac{Q}{2\pi c^2} G(p)\, p K_0\left(\frac{r}{c}\sqrt{p^2+4\omega^2}\right),\\ V_r(r,p) &= \frac{Qg}{2\pi c^2} G(p) \frac{p^2}{\sqrt{p^2+4\omega^2}} K_1\left(\frac{r}{c}\sqrt{p^2+4\omega^2}\right),\\ V_s(r,p) &= -\frac{Qg\omega}{\pi c^3} G(p) \frac{p}{\sqrt{p^2+4\omega^2}} K_1\left(\frac{r}{c}\sqrt{p^2+4\omega^2}\right).\end{aligned} \tag{4}$$

Here p is the transform parameter; $K_0(z)$ and $K_1(z)$ are modified Hankel functions of zero and first order, respectively;

$$G(p) = f(p)\left[1 + \frac{4\omega^2}{p^2}\right]$$

and is the image of the original

$$G(t) = f(t) + 4\omega^2 \int_0^t (t-\xi) f(\xi)\, d\xi.$$

Passing from the images (4) to the originals, with the aid of the convolution theorem we find, for $t \geq r/c$,

$$\zeta(r,t) = \frac{Q}{2\pi c^2}\frac{\partial}{\partial t}\int_0^{t-r/c} \frac{\cos\left[2\omega\sqrt{(t-\xi)^2-(r/c)^2}\right]}{\sqrt{(t-\xi)^2-(r/c)^2}} G(\xi)\, d\xi; \tag{5}$$

$$V_r(r,t) = \frac{Qg}{2\pi c^2 r}\frac{\partial}{\partial r}\int_0^{t-r/c} \frac{\cos\left[2\omega\sqrt{(t-\xi)^2-(r/c)^2}\right]}{\sqrt{(t-\xi)^2-(r/c)^2}} (t-\xi) G(\xi)\, d\xi; \tag{6}$$

$$V_s(r,t) = -\frac{Qg}{2\pi c^2 r}\frac{\partial}{\partial t}\int_0^{t-r/c} \sin\left[2\omega\sqrt{(t-\xi)^2-(r/c)^2}\right] G(\xi)\, d\xi. \tag{7}$$

If $t < r/c$, then $\zeta(r,t) = V_r(r,t) = V_s(r,t) = 0$.

Formulas (5)–(7) give the exact solution of the problem posed. If $\omega = 0$, then the solution (5)–(7) passes into the known solution of the problem of ring waves in a stationary basin (^{2}), with the tangential component of the velocity becoming zero.

Suppose that $f(0) = 0$; then the elevation of the free surface of the fluid (5) can be rewritten in the form $\zeta(r,t) = \zeta_1(r,t) + \zeta_2(r,t)$, where

$$\zeta_1 = \frac{Q}{2\pi c^2}\int_0^{t-r/c} \frac{\cos\left[2\omega\sqrt{(t-\xi)^2-(r/c)^2}\right]}{\sqrt{(t-\xi)^2-(r/c)^2}} \frac{df(\xi)}{d\xi}\, d\xi; \tag{8}$$

$$\zeta_2 = \frac{2Q\omega^2}{\pi c^2} \int_0^{t-r/c} \frac{\cos\left[2\omega\sqrt{(t-\xi)^2-(r/c)^2}\right]}{\sqrt{(t-\xi)^2-(r/c)^2}}\, \varphi(\xi)\, d\xi, \tag{9}$$

where

$$\varphi(t) = \int_0^t f(\xi)\, d\xi, \tag{10}$$

and, consequently, $Q\varphi(t)$ is the total discharge of the source over the time t. Formula (8) represents the elevation determined by the time derivative of the source discharge. If $\omega = 0$, then ζ_1 exactly coincides with the elevation of the free surface in a stationary basin. The term ζ_2, determined by formula (9), is essentially due to rotation. If the basin is stationary, it becomes zero. The additional elevation ζ_2, determined by the integral of the source discharge, is caused by the change in the total discharge of the source during the time interval $t - r/c$.

Following Lamb (^{2}), it is easy to show that the action of a point source is equivalent to the action of a disturbing concentrated pressure force $P(t)$, this force being related to the source discharge by the formula

$$P(t) = \frac{Q\rho c^2}{h} \int_0^t f(\xi)\, d\xi. \tag{11}$$

As is known, if the basin is stationary, then waves on the surface of the fluid are formed only when the second time derivative of the function describing the total disturbing pressure force is nonzero. As a result of the rotation of the basin, in addition to this elevation, due to $\delta^2 P(t)/\delta t^2$, there appears an additional elevation caused by the force $P(t)$ itself.

Taking the small quantity $\Delta t = t - r/c$, we obtain an approximate expression for the leading front of the wave

$$\zeta(r,t) \simeq \frac{Q}{\pi c}\, \Delta t\, [f'(\Delta t) + 4\omega^2 \varphi(\Delta t)].$$

Thus, the leading front of the wave is composed of two terms. The first term is determined by the derivative of the source discharge and does not depend on the rate of rotation of the basin; the second term is determined by the total discharge of the source over the time interval Δt and is entirely due to the rotation of the basin.

Suppose now that the source acted according to an arbitrary law during a finite time interval T, and then was switched off. Let us find the limit of expressions (5)–(7) as $t \to \infty$.

$$\zeta^* = \lim_{t\to\infty} \zeta(r,t) = \frac{2Q\omega^2}{\pi c^2}\,\varphi(T)\,K_0\left(\frac{r}{c}\,2\omega\right); \tag{12}$$

$$V_r^* = \lim_{t\to\infty} V_r(r,t) = 0,$$

$$V_s^* = \lim_{t\to\infty} V_s(r,t) = -\frac{2Qg\omega^2}{\pi c^3}\,\varphi(T)\,K_1\left(\frac{r}{c}\,2\omega\right). \tag{13}$$

It follows from formulas (12) and (13) that a permanent elevation, independent of time, may arise on the free surface of the fluid, while the motion of the fluid takes place along concentric circles with center at the origin of coordinates. Such motions (3) are called geostrophic; for them the elevation and velocity are related by

$$V_s = \frac{g}{2\omega}\,\frac{\partial\zeta}{\partial r},$$

which is obtained from equations (1), if it is assumed that the functions ζ, V_r, and V_s do not depend on time. It is evident that the geostrophic elevation ζ^* and velocity V_s^*, found by the limiting transition (12), (13), satisfy this relation. From formulas (12) and (13) it follows that a necessary and sufficient condition for the occurrence of geostrophic motion is the fulfillment of the inequality

$$\varphi(T) \neq 0, \tag{14}$$

where $\varphi(T)$ is determined by formula (10) and, up to the constant factor Q, represents the total discharge of the source over the entire time of its action. In this case the geostrophic elevations and velocity do not depend on the law according to which the source acted.

Bearing in mind the relation of the source discharge to the disturbing concentrated pressure force (11), we find that inequality (14) is equivalent to the condition $P(T) \neq 0$. Consequently, a necessary and sufficient condition for the appearance of a geostrophic elevation is the presence of a constant component of the disturbing pressure force for all t such that $T < t < \infty$. This component is precisely the force that maintains the geostrophic motion.

Let us consider the establishment of a periodic regime. Here two cases are possible: 1) $f(t) = \cos\sigma t$; 2) $f(t) = \sin\sigma t$. According to formula (11), the disturbing concentrated pressure force in the first case is equal to

$$P(t) = \frac{Q\rho c^2}{\sigma h}\sin\sigma t,$$

and in the second

$$P(t) = -\frac{Q\rho c^2}{\sigma h}\cos\sigma t + \frac{Q\rho c^2}{\sigma h}. \tag{15}$$

In the second case there is a constant component of the pressure force, and therefore the appearance of geostrophic motion should be expected. We note that the condition for the occurrence of geostrophic motion obtained above cannot be transferred to the case under consideration, that of establishing a periodic regime, since in deriving that condition it was assumed that the source acts over a finite interval of time. In the present case the source operates for an infinitely long time. This difficulty can be removed in the following way. It is easy to show that the application of a harmonic disturbing force does not produce geostrophic elevation or velocity. Since the problem is linear, the motion caused by the disturbing force (15) may be considered as the sum of motions caused by a harmonic disturbing force and by a constant disturbing force, which are applied at the instant $t = 0$. But it was shown earlier that the application of a constant disturbing force produces geostrophic elevation and velocity. Consequently, the switching on of a source with discharge $Q\sin\sigma t$ produces the appearance of geostrophic motion, whereas the switching on of a source with discharge $Q\cos\sigma t$ does not produce this motion.

These conclusions are not difficult to obtain by direct calculation. Putting $f(t) = e^{i\sigma t}$ and using formula (5), we find the limiting elevation of the free surface as $t \to \infty$

$$\zeta^* = \frac{Q}{4c^2}\frac{\sigma^2 - 4\omega^2}{\sigma}e^{i\sigma t}H_0^{(2)}\left(\frac{r}{c}\sqrt{\sigma^2 - 4\omega^2}\right) + i\frac{2Q\omega^2}{\pi c^2\sigma}K_0\left(\frac{r}{c}2\omega\right), \tag{16}$$

if $\sigma > 2\omega$, and

$$\zeta^* = -\frac{iQ}{2\pi c^2}\frac{4\omega^2 - \sigma^2}{\sigma}e^{i\sigma t}K_0\left(\frac{r}{c}\sqrt{4\omega^2 - \sigma^2}\right) + i\frac{2Q\omega^2}{\pi c^2\sigma}K_0\left(\frac{r}{c}2\omega\right), \tag{17}$$

if $\sigma < 2\omega$. Here $H_0^{(2)}(z)$ is the Hankel function of the second kind of zero order. The real parts of formulas (16), (17) determine the elevation caused by a source with discharge $Q\cos\sigma t$, the imaginary parts—by a discharge $Q\sin\sigma t$. The second terms in these formulas describe the geostrophic elevation, which appears when the source is switched on according to a sinusoidal law. In this case the progressive wave (16) or the standing wave (17) is superposed on the geostrophic elevation.

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Note: Figure translations are in progress. See original paper for figures.

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