CONSTRUCTION OF AN EFFECTIVELY UNATTAINABLE CARDINAL IN A NATURAL EXTENSION OF THE ZERMELO-FRAENKEL SYSTEM

MATHEMATICS

1969

SovietRxiv

Abstract

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UDC 519.513/.514

MATHEMATICS

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CONSTRUCTION OF AN EFFECTIVELY UNATTAINABLE CARDINAL IN A NATURAL EXTENSION OF THE ZERMELO-FRAENKEL SYSTEM

(Presented by Academician P. S. Novikov on 26 XII 1968)

A natural extension of the Zermelo–Fraenkel system with or without the axiom of choice is considered, see ([1]) (denotation ZF), obtained by adding to the syntax of ZF a two-place relation $\mathrm{Tr}(x, y)$ and a number of axioms such that $\mathrm{Tr}(x, y)$ can be interpreted as the predicate: "the formula x, under valuation y, is true." If the system ZF with the axiom on the existence of an unattainable cardinal is consistent, then the system $ZF\,\mathrm{Tr}$ thus obtained is also consistent.

In this extension of ZF one constructs a cardinal a_0, whose cofinality is ω_0, such that

$$\forall a\colon \mathrm{Ord}(a), a < a_0 :\rightarrow: \overline{P}a < a_0$$

and the ordinal a_0 is not named by any term of ZF in which any sets of ranks less than a_0 are taken as constants. Moreover, no sequence of ordinals cofinal in a_0 can be named in this way either. In this sense a_0 may be called an effectively unattainable cardinal. The cardinal a_0 can be included as an element in some standard transitive model of ZF, and it remains unnamed "inside" this model. The cardinal a_0 refutes the conceivable hypothesis that for every cardinal whose cofinality is ω_0 there exists a term of ZF and such sets (of ranks smaller than this cardinal) as constants of this term that the term "names" this cardinal.

The proof follows naturally from the possibility of constructing in ZF an absolute (relative to the universe) natural model of ZF.

Formalization of the syntax of ZF by means of ZF. In what follows we shall regard the systems ZF and $ZF\,\mathrm{Tr}$ as supplemented with Hilbert ι-terms in the generally accepted way. $\dot{x}, \dot{y}, \dot{M}, \dot{x}_1, \dots, \dot{x}_n, \dots$ are constant terms satisfying the formula expressing "to be a variable." φ is a special variable ranging over objects satisfying the formula expressing "to be a formula." $t(x, n)$ is a formula expressing "x is a term with n variables." $\exists T \dot{x}_0 \dots \dot{x}_n$ and $\forall T \dot{x}_0 \dots \dot{x}_n$

are abbreviations; for example, the first of them should be deciphered as follows: “there exists an x such that $t(x, n)$ and the variables in x are the first n variables in the series $\dot{x}_0 \dots \dot{x}_n \dots$,” i.e. the existence is asserted of a term T with variables $\dot{x}_0 \dots \dot{x}_n$. $\exists\langle x_0 \dots x_n\rangle$ and $\forall\langle x_0 \dots x_n\rangle$ are abbreviations; for example, the first of them is deciphered as follows: “there exists a tuple $\langle x_0 \dots x_n\rangle$.” $Ak_{ZF}(x)$ is the formula “saying” “x is an axiom of ZF.” O is a special variable for a formula expressing: “to be a function defined on the set of all variables” (valuation).

$$\begin{pmatrix} \dot{y}_1 \dots \dot{y}_n \\ x_1 \dots x_n \end{pmatrix}$$

is a term (for fixed n) which, from the tuple $\langle x_1 \dots x_n\rangle$, yields the set of pairs $\{\langle \dot{y}_1 x_1\rangle \dots \langle \dot{y}_n x_n\rangle\}$ (special valuation). xy is a two-place term; if x is a valuation and y is a special valuation of the form

$$\begin{pmatrix} y_1 \dots y_n \\ x_1 \dots x_n \end{pmatrix},$$

then its value is such a valuation that its values on the

$\dot{y}_1 \dots \dot{y}_n$ are $x_1 \dots x_n$, and on the other variables are the same as for y in x. We shall use this term when, instead of x, an arbitrary valuation O is substituted, and instead of y a certain concrete special valuation

$$\begin{pmatrix} \dot{y}_1 \dots \dot{y}_n \\ x_1 \dots x_n \end{pmatrix}$$

(x, y) is also a two-place term of ZF Tr such that if x is a term with variables $\dot{x}_1 \dots \dot{x}_n$ (denote it by $T\dot{x}_1 \dots \dot{x}_n$), and y is a special valuation for these variables

$$\begin{pmatrix} \dot{x}_1 \dots \dot{x}_n \\ x_1 \dots x_n \end{pmatrix},$$

then the value of the term is such a set z that

$$Tr\left(\dot{x} = T\dot{x}_1 \dots \dot{x}_n, \begin{pmatrix} \dot{x}x_1 \dots \dot{x}_n \\ xx_1 \dots x_n \end{pmatrix}\right). \lim_z T\dot{x}_0 \dots \dot{x}_n, \begin{pmatrix} \dot{x}_1 \dots \dot{x}_n \\ x_1 \dots x_n \end{pmatrix}$$

is a three-place term whose value is the least upper bound of the ranks of the elements

$$\left\{T\dot{x}_0 \dots \dot{x}_n, \begin{pmatrix} \dot{x}_0 \dots \dot{x}_n \\ x_0 \dots x_n \end{pmatrix} \,\middle|\, x_0 \in z\right\}.$$

$x'y$ is a two-place term such that if x is a valuation and y is a formula with free variables $\dot{y}_1 \dots \dot{y}_n$, then the value of the term is the set of the values of the valuation x on $\dot{y}_1 \dots \dot{y}_n$.

x_y is a two-place term; if x is a formula and y is a variable, then the value of this term is the formula obtained from x by relativizing all variables bound in x to y.

$\mathrm{cf}(\alpha)$ is a term which gives, for the ordinal α, the least ordinal cofinal with α. $\mathrm{cf}(\gamma, \alpha)$ is a formula saying "γ is cofinal with α." $\mathrm{Ist}_M(x, y)$ is a ZF formula defining truth on the set M for the formulas x under valuations y from elements of M.

In the formulas $Tr(x, y)$ and $\mathrm{Ist}_M(x, y)$, if x is a closed formula, then the argument y is omitted: $Tr(x)$ and $\mathrm{Ist}_M(x)$. The system $ZF\ Tr$ is obtained from the system ZF by extending the syntax of the latter with the new relation $Tr(x, y)$ (the definition of formula is changed correspondingly). The axioms of $ZF\ Tr$ are the axioms of ZF, with the axiom schema of substitution strengthened so that it applies not only to all formulas of ZF, but also to formulas of $ZF\ Tr$, and the following 7 axioms (syntactic axioms of $ZF\ Tr$):

$$Tr\left(\dot{x} \in \dot{y},\ \begin{pmatrix} \dot{x} & \dot{y} \\ x & y \end{pmatrix}\right) \equiv x \in y,$$

$$Tr(\dot{\neg}\varphi, O) \equiv \neg Tr(\varphi, O), \qquad Tr(\varphi \& \psi, O) \equiv Tr(\varphi, O) \& Tr(\psi, O),$$

$$Tr(\dot{\exists}\dot{x}\varphi\dot{x}, O) \equiv \exists x\, Tr\left(\varphi\dot{x},\ O\begin{pmatrix} \dot{x} \\ x \end{pmatrix}\right)$$

and so on.

A cardinal α is called inaccessible if it has the following property:

$$\mathrm{cf}(\alpha) = \alpha \ \&\ \forall \alpha'.\, \alpha' < \alpha \to \overline{\overline{P\alpha'}} < \alpha.$$

Let us consider two ways of naming an ordinal by ZF terms:

$$H_{\lim}(\alpha) \Leftrightarrow \exists n \exists T \dot{x}_0 \dots \dot{x}_n \exists \langle x_1 \dots x_n \rangle \exists \gamma.\ \mathrm{cf}(\gamma, \alpha)\ \&\ \lim_{\{\beta | \beta < \gamma\}} T \dot{x}_0 \dots \dot{x}_n,$$

$$\begin{pmatrix} \dot{x}_0 \dots \dot{x}_n \\ \beta \dots x_n \end{pmatrix} = \alpha,$$

$$H(\alpha) \Leftrightarrow \exists n \exists T \dot{x}_1 \dots \dot{x}_n \exists \langle x_1 \dots x_n \rangle\, T \dot{x}_1 \dots \dot{x}_n,\ \begin{pmatrix} \dot{x}_1 \dots \dot{x}_n \\ x_1 \dots x_n \end{pmatrix} = \alpha.$$

An ordinal α is called effectively inaccessible if

$$\neg H(\alpha) \;\&\; \neg H_{\lim}(\alpha)$$

and

$$\forall \alpha'.\, \alpha' < \alpha \to \overline{\overline{P\alpha'}} < \alpha.$$

Denote the conjunction of the last three formulas by $In(\alpha)$. By a model here we always mean a standard and transitive model (see ([2])). If a model, as a set, has the form

$$\bigcup_{\beta<\alpha} R_\beta, \quad \text{where} \quad R_\beta = \bigcup_{\gamma<\beta} P(R_\gamma),$$

it is called natural. A model M is called absolute if

$$\forall \varphi \forall O.\, O'\varphi \subseteq M \to Tr(\varphi, O) \equiv Tr\left(\varphi_M,\, O\begin{pmatrix}\dot{M}\\ M\end{pmatrix}\right).$$

As will follow from the proof of Lemma 2, absolute natural models form a class and, consequently, can be numbered (in increasing order with respect to $\subseteq$) by all ordinals: $M_0, \dots, M_\beta, \dots$. The least ordinal not belonging to M_β will be denoted by α_β and called the boundary of the model M_β. Let $M_\alpha, M_\beta, M_\gamma$, etc. be special variables for objects satisfying the formula expressing "to be an absolute natural model of ZF," and let $\alpha_\beta, \alpha_\gamma$, etc. be special variables for objects satisfying the formula expressing "to be the boundary of an absolute natural model of ZF." $M_0, M_1, M_2, \dots$, etc. will be used as designations of constant terms that "define" the models $M_0, M_1, M_2, \dots$, etc. (the possibility of constructing such terms of ZF$'$Tr follows from Lemma 2), and $\alpha_0, \alpha_1, \dots, \alpha_\omega$ as designations of constant terms naming the boundaries of the models $M_0, M_1, \dots, M_\omega$.

Proposition 1. If the system ZF with the additional axiom on the existence of an inaccessible cardinal is consistent, then the system ZF$'$Tr is also consistent.

Proposition 2. It is consistent relative to ZF$'$Tr that if α is the boundary of a natural model of ZF, then $\mathrm{cf}(\alpha) < \alpha$.

Proposition 3. If β is of the second kind, then

$$M_\beta = \bigcup_{\gamma<\beta} M_\gamma.$$

If β is of the first kind, then M_β is obtained by applying the process described in Lemma 2 to the set $M_{\beta-1}$. $\mathrm{cf}(\alpha_\beta) = \omega_0$ if $\beta < \omega_1$. $\mathrm{cf}(\alpha_\beta) \geq \mathrm{cf}(\beta)$.

Theorem. One can name by terms of $\mathrm{ZF}'\mathrm{Tr}$ such ordinals α_0 and α_1 that they will be the boundaries of absolute natural models of ZF, M_0 and M_1 respectively, and $\alpha_0 \in M_1$, $\mathrm{cf}(\alpha_0) = \omega_0$, $\mathrm{cf}(\alpha_1) = \omega_0$, and $\mathrm{In}(\alpha_0)$, $(\mathrm{In}(\alpha_0))_{M_1}$.

The proof follows from the lemmas.

Lemma 1.

$$\underset{\mathrm{ZF\,Tr}}{\vdash} \forall\varphi \ \mathrm{Ak}_{\mathrm{ZF}}(\varphi) \to \mathrm{Tr}(\varphi).$$

Lemma 2.

$$\underset{\mathrm{ZF}'\mathrm{Tr}}{\vdash} \exists M\alpha: \ M_\alpha = \bigcup_{\beta<\alpha} R_\beta \ \& \ \mathrm{cf}(\alpha) = \omega_0 \ \& \ \forall\varphi\,\forall O\,O'\varphi \subseteq M \to$$

$$\to \mathrm{Tr}\left(\varphi_{\dot M} \equiv \varphi, \ O\binom{\dot M}{M}\right).$$

We number all terms $t_1 \dots t_n \dots$. For some ordinal γ_0 we form the set of values of the term t_1 on the set

$$\bigcup_{\beta<\gamma_0} R_\beta$$

and denote the least upper bound of this set by γ_1; consider the union of the sets of values of the terms t_1 and t_2 on the set

$$\bigcup_{\beta<\gamma_1} R_\beta$$

and denote the least upper bound of this set by γ_2, and in this way we construct $\gamma_3 \dots \gamma_n \dots$. The ordinal γ is $\sup_n\{\gamma_n\}$. The set

$$\bigcup_{\beta<\gamma} R_\beta$$

is the set required in the lemma.

Lemma 3.

$$\underset{\mathrm{ZF}'\mathrm{Tr}}{\vdash} \forall\varphi \ \mathrm{Ak}_{\mathrm{ZF}}(\varphi) \to \mathrm{Ist}_{M_1}(\varphi).$$

Lemma 4.

$$\underset{\mathrm{ZF}'\mathrm{Tr}}{\vdash} \neg H(\alpha_0), \qquad \underset{\mathrm{ZF}'\mathrm{Tr}}{\vdash} \mathrm{In}(\alpha_0), \qquad \underset{\mathrm{ZF}'\mathrm{Tr}}{\vdash} (\mathrm{In}(\alpha_0))_{M_1}.$$

Corollary 1. If ZF is consistent, then the following formula can be adjoined to ZF without contradiction:

$$\exists M.\ M = \bigcup_{\beta<\alpha} R_\beta \ \& \ \forall\varphi \ \mathrm{Ak}_{\mathrm{ZF}}(\varphi) \to \mathrm{Ist}_M\,\varphi \ \& \ \exists a.\ a \in M \ \& \ \mathrm{In}_M(a),$$

where $\text{In}_M(a)$ is the formula obtained from the formula $\text{In}(a)$ by replacing in it every occurrence of the formula $\text{Tr}(x, y)$ by the formula $\text{Ist}_M(x, y)$.

Corollary 2. If ZF is consistent, then it is consistent to extend it in the following way: extend the syntax of ZF by a new constant α_0 and add the axioms: 1) $\text{card}(\alpha_0)$, $\text{cf}(\alpha_0) = \omega_0$; 2) $\forall \alpha'. \alpha' < \alpha \to P\alpha' < \alpha_0$; 3_n) a list of axioms of the following form:

$$\neg\exists x_1 \dots x_n' \,(\text{for all } x_i < \alpha_0 \;\&\; T(x_1 \dots x_n) = \alpha_0).$$

where T is an arbitrary term of ZF.

Corollary 3. For every a_β there exist a_β (in cardinality) distinct natural models of ZF preceding (in the sense of the relation $\equiv$) M_β.

All M_β are elementarily equivalent to one another and to the universe ZF Tr (i.e., the set of closed formulas true in M_β coincides with the set $\{\varphi \mid \text{Tr}(\varphi)\}$). As Corollary 3 shows, before M_β (in the sense of $\equiv$) there are many other natural models of ZF. Are there among them (and how many?) such natural models that are elementarily equivalent to M_β?

Corollary 4. For every a_β there exist a_β (in cardinality) distinct natural models of ZF ordered by the relation $\equiv$ (all of them $\equiv M_\beta$), and each of them is an elementary extension of any preceding one and is elementarily equivalent to M_β with respect to formulas of ZF, and their ordinal numbers are cofinal in ω_0.

Remark. Many of the assertions made, although formulated here for absolute natural models, are in fact also true for natural models, and even simply for sets of the form $\bigcup_{\beta<\alpha} R_\beta$. The same consideration could also have been carried out in a "weaker" system than ZF Tr, for example in an extension of ZF obtained by adding the axiom of the existence of some model of ZF of a special kind or of the existence of an inaccessible cardinal, etc.

The author does not know whether there exists a cardinal a such that $\neg H_{\text{lim}}(a)$ and $H(a)$, $\text{cf}(a) = \omega_0$, and $\forall a'. a' < a \to \overline{Pa'} < a$.

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Received
12 XII 1968

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Note: Figure translations are in progress. See original paper for figures.