PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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Abstract

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MATHEMATICS

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PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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Consider the differential equation

$$dx/dt = f(t, x), \tag{1}$$

where $x = x(t)$ is the unknown function with values in a certain complex Banach space E; $f(t, x)$ is an operator which, for each fixed t, acts in E and is periodic in t with period 2π. It is assumed that the operator $f(t, x)$ maps a strongly continuous 2π-periodic function $x(t)$ into the likewise strongly continuous function $f(t, x(t))$.

There are various methods for studying periodic solutions of equation (1), for example, ([1,2]). In the present work one more method is proposed for solving the problem of the existence and finding of periodic solutions of equation (1), which is close to Cesari' s method ([2]).

Consider the operator B_1, defined by the equality

$$B_1 x(t) = a + \frac{1}{2\pi} \int_t^{t+2\pi} (s - t - 2\pi) f(s, x(s))\, ds.$$

Let X be the set consisting of strongly continuous 2π-periodic functions. Then the operator B_1 acts in X for any value of the parameter $a \in E$.

Lemma. *Let $x(t, a)$ be a fixed point of the operator B_1. If*

$$\int_0^{2\pi} f(s, x(s, a))\, ds = 0, \tag{2}$$

then the function $x(t, a)$ is a 2π-periodic solution of equation (1).

The operator B_1 is convenient for studying the equation

$$dx/dt = A(t)x + f(t), \tag{3}$$

where $A(t)$ is a linear bounded operator, continuous and periodic in t with period 2π, and $f(t)$ is a continuous 2π-periodic function.

Introduce the following series:

$$U(t) = -\frac{1}{2\pi}\int_{t+2\pi}^{t} A(s)\,ds + \frac{1}{2\pi}\int_{t+2\pi}^{t}\left(\int_{t+2\pi}^{s} A(s)\,ds\right) A(s)\,ds - \dots,$$

$$g(t) = \frac{1}{2\pi}\int_{t+2\pi}^{t} f(s)\,ds - \frac{1}{2\pi}\int_{t+2\pi}^{t}\left(\int_{t+2\pi}^{s} A(s)\,ds\right) f(s)\,ds + \dots$$

In this case we have

$$U(t) = \frac{1}{2\pi}V(t,0)(V(2\pi,0) - I)V(0,t),$$

where I is the identity operator, and $V(t,\tau)$ is the solution of the integral equation

$$V(t,\tau) = I + \int_{\tau}^{t} V(t,s)A(s)\,ds.$$

Theorem 1. *If equation (3) has a 2π-periodic solution $x(t)$, then the equality $U(t)x(t) + g(t) = 0$ holds.*

The proof is carried out using the operator B_1.

One can give the following sufficient condition for the invertibility of the operator $U(t)$.

Theorem 2. Suppose

$$\alpha = \|A(0)\|, \qquad \gamma = \sup_t \|A(t)\|, \qquad \beta = \frac{1}{2\pi}\int_0^{2\pi} \|A(s) - A(0)\|\,ds.$$

Suppose the operator $I - 2\pi A(0)$ is invertible and

$$\left\|(I - 2\pi A(0))^{-1}\right\| = (1 + 2\pi\rho)^{-1}.$$

Then, if

$$\frac{1 + 2\pi\beta\exp(2\pi\gamma)}{1 + 2\pi\rho - 4\pi^2\alpha\gamma\exp(2\pi\gamma)} < 1,$$

equation (3) has one and only one 2π-periodic solution.

We now introduce a more general operator B_2, where

$$B_2 y(t) = -\varphi(t) + \frac{1}{2\pi}\int_t^{t+2\pi} (s - t - \pi) f(s, a_0 + \varphi(s) + y(s))\, ds,$$

$$\varphi(t) = \sum_{0<|n|\le k} a_n e^{int}.$$

The operator B_2, for arbitrary values of the parameters $a_0, a_1, a_{-1}, \dots, a_k, a_{-k}$, also acts in X. If $y(t, a_0, \dots, a_{-k})$ is a fixed point of this operator, then $x(t) = a_0 + \varphi(t) + y(t, a_0, \dots, a_{-k})$ is a 2π-periodic solution of equation (1), when the parameters satisfy equation (2).

Theorem 3. Suppose the operator $f(t, x)$ satisfies the Lipschitz condition

$$\|f(t, x') - f(t, x'')\| \le p(t)\|x' - x''\|, \qquad x', x'' \in E. \tag{4}$$

Suppose $H(t)$, for each fixed t, is a linear bounded operator, continuous and 2π-periodic in t, and satisfying the condition

$$\|H(t)(x' - x'') - f(t, x') + f(t, x'')\| \le q(t)\|x' - x''\|, \qquad x', x'' \in E. \tag{5}$$

Here $p(t)$ and $q(t)$ are certain positive 2π-periodic functions. In addition, suppose that for the operator $\int_0^{2\pi} H(t)\, dt$ there exists a bounded inverse, and that the inequalities

$$\mu = \sup_t \frac{1}{2\pi}\int_{-\pi}^{\pi} |s|\, p(s + t + \pi)\, ds < 1,$$

$$\frac{1}{1-\mu}\left\|\left(\int_0^{2\pi} H(t)\, dt\right)^{-1}\right\|\int_0^{2\pi} (q(t) + \mu\|H^*(t)\|)\, dt < 1,$$

hold, where

$$H^*(t) = H(t) - \frac{1}{2\pi}\int_0^{2\pi} H(t)\, dt.$$

Then equation (1) has one and only one 2π-periodic solution.

Proof. The operator B_2 with one parameter a_0, i.e., with $\varphi(t) \equiv 0$, is, by (4), a contraction operator. Further, taking into account the identity

$$\int_0^{2\pi} H(t)\,dt\,a_0 = \int_0^{2\pi} (H(t)x(t) - f(t,x(t)))\,dt - \int_0^{2\pi} H^*(t)y(t,a_0)\,dt +$$

$$+ \int_0^{2\pi} f(t,x(t))\,dt,$$

where $x(t) = a_0 + y(t,a_0)$, $y(t,a_0)$ is a fixed point of B_2, the operator is constructed

$$Fa_0 = \left(\int_0^{2\pi} H(t)\,dt\right)^{-1} \left[\int_0^{2\pi} (H(t)x(t) - f(t,x(t)))\,dt - \int_0^{2\pi} H^*(t)y(t,a_0)\,dt\right].$$

By virtue of (5), the operator F has a fixed point a_0^*. Then the function $x^*(t) = a_0^* + y(t,a_0^*)$ is a periodic solution of equation (1). The theorem is proved.

By a somewhat different method one can obtain a proposition for equation (3).

Theorem 4. *Let the operator* $\int_0^{2\pi} A(s)\,ds$ *have a bounded inverse with norm* η, *and*

$$M = \sup_t \frac{1}{2\pi} \int_{-\pi}^{\pi} \|sA(s+t+\pi)\|\,ds, \qquad N = \int_0^{2\pi} \|A(s)\|\,ds.$$

Then, if $M(1+\eta N) < 1$, *equation (3) has a* 2π*-periodic solution.*

Proof is carried out by the method of successive approximations, putting

$$y_{n+1}(t) = \frac{1}{2\pi} \int_t^{t+2\pi} (s-t-\pi)[A(s)(a_n + y_n(s)) + f(s)]\,ds,$$

$$\int_0^{2\pi} A(s)(a_n + y_n(s))\,ds + \int_0^{2\pi} f(s)\,ds = 0.$$

The following proposition is, in a certain sense, a generalization of Theorem 3.

Theorem 5. *Let* E *be a Banach space such that for every function* $x(t) \in X$ *the equality*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \|x(t)\|^2\,dt = \sum_{n=-\infty}^{\infty} \|a_n\|^2, \qquad a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t)e^{-int}\,dt. \tag{6}$$

holds. Let the operator $f(t,x)$ satisfy condition (4), where $p(t) = p$ is constant. Let there exist a constant linear operator H satisfying condition (5), and suppose that the eigenvalues of the operator H are distinct from numbers of the form ni, $n = 0, \pm 1, \ldots, \pm k$, where k is some natural number. Then, if

$$\mu_k = 2p^2 \sum_{n=k+1}^{\infty} \frac{1}{n^2} < 1, \tag{7}$$

$$\frac{1}{2\pi(1-\mu_k)} \int_0^{2\pi} q^2(t)\,dt \sum_{|n|\le k} \|(niI - H)^{-1}\|^2 < 1, \tag{8}$$

then equation (1) has one and only one 2π-periodic solution.

Proof. In the metric space of sequences of the form $\{a_n\}$, $n = \pm(k+1), \pm(k+2), \ldots$, $a_n \in E$, $\sum \|a_n\| < +\infty$, an operator Φ is constructed, defined by the right-hand sides of the equalities

$$a_n = \frac{1}{2\pi n i} \int_{-\pi}^{\pi} f(t, a_0 + \varphi(t) + y(t)) e^{-int}\,dt, \qquad |n| > k,$$

where

$$y(t) = \sum_{|n|>k} a_n e^{int}.$$

The operator Φ, by virtue of (7), has a fixed point $\{a_n^*\}$, the coordinates of which depend on the parameters. Next, in the parameter space an operator Ψ is introduced, defined by the right-hand sides of the equalities

$$a_n = \frac{1}{2\pi}(-niI + H)^{-1} \int_{-\pi}^{\pi} (Hx(t) - f(t, x(t))) e^{-int}\,dt, \qquad |n| \le k,$$

where

$$x(t) = a_0 + \varphi(t) + y^*(t), \qquad y^*(t) = \sum_{|n|>k} a_n^* e^{int}.$$

By virtue of (8), the operator Ψ has a fixed point with coordinates a_n^*, $|n| \le k$. Then, in turn, the function $y^*(t)$, for $a_n = a_n^*$, $|n| \le k$, is a fixed point of the operator B_2. The theorem is proved.

We note that for the linear equation (2), when A is a constant matrix, the conditions of Theorem 5 are at the same time necessary. Indeed, in this case it is necessary that the eigenvalues of the matrix A be distinct from numbers of the form ni, $n = 0, \pm 1, \pm 2, \dots$. Then the role of the operator H is played by the matrix A. In this case condition (8) holds for any k, and condition (7) for sufficiently large k. We also note that Theorem 5 can be formulated using a variable operator $H(t)$.

The preceding method can be generalized by considering equations of higher order. For example, for the equation

$$d^2x/dt^2 = f(t, x), \tag{9}$$

where $f(t, x)$ is the same operator as in equation (1), the following holds.

Theorem 6. *Let the operator $f(t, x)$ satisfy condition (4). Let there exist an operator $H(t)$, satisfying condition (5), such that the eigenvalues of the operator*

$$\int_0^{2\pi} H(t)\, dt$$

are distinct from numbers of the form $-2\pi n^2$, $n = 0, 1, \dots, k$, where k is some natural number or zero. Then, if

$$\mu_k = \frac{1}{\pi} \int_0^{2\pi} p(t)\, dt \sum_{n=k+1}^{\infty} \frac{1}{n^2} < 1,$$

$$\frac{1}{1-\mu_k} \sum_{|n| \le k} \left\| \left(2\pi n^2 I + \int_0^{2\pi} H(t)\, dt \right)^{-1} \right\| \int_0^{2\pi} (q(t) + \|H^*(t)\|)\, dt < 1,$$

then equation (9) has one and only one 2π-periodic solution.

Proof is carried out using the operator

$$B_3 y(t) = -\varphi(t) + \frac{1}{4\pi^2} \int_t^{t+2\pi} (s-t-\pi) \int_s^{s+2\pi} (\tau-s-\pi) f(\tau, a_0 + \varphi(\tau) + y(\tau))\, d\tau.$$

We note that if, for any $x(t) \in X$, equality (7) holds, then under the conditions of Theorem 6 the estimates can, in a certain sense, be improved.

Remark. The method presented can without particular difficulty be extended to equations of the form (3)

$$dx/dt = \omega(t, \tilde{x}),$$

where $\tilde{x} = x(t)$, $\omega(t, \tilde{x})$ is an operator with values in E, defined for each t on the set X and periodic in t with period 2π, i.e., the identity $\omega(t + 2\pi, \tilde{x}) = \omega(t, \tilde{x})$ holds for all t and any fixed $\tilde{x} \in X$.

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Note: Figure translations are in progress. See original paper for figures.