ON STRONGLY CLOSED MAPPINGS

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Abstract

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V. V. FEDORCHUK

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Strongly closed mappings were introduced in ([3]). There it was proved that such mappings do not lower dimension. Somewhat earlier, a strongly closed mapping was used by me in the construction of a bicompactum with the first axiom of countability which cannot be partitioned by a set of smaller dimension ([2]).

In the present note the intrinsic properties of the class of strongly closed mappings are studied. From this point of view, the consideration of bicompact and irreducible strongly closed mappings turns out to be the most substantial. Every space* has a maximal preimage with respect to the class of such mappings. Bicompact irreducible strongly closed mappings onto normal spaces admit a good constructive description.

§ 1. **Definition.** A continuous mapping $f : X \to Y$ is called **strongly closed** if, for every point $y \in Y$ and every finite covering $\{U_i \mid i = 1, \dots, j\}$ of its preimage $f^{-1}y$ by sets open in X, the set

$$\{y\} \cup \left(\bigcup_{i=1}^{j} f^{\#} U_i \right)$$

is open in Y.

It follows from the definition that if the set $Y \setminus fX$ is nonempty, then it consists of isolated points of the space Y, and the set fX is both open and closed in Y. Strongly closed mappings possess a number of properties inherent in all closed mappings.

Lemma 1. Let $f : X \to Y$ and $g : Y \to Z$ be continuous mappings such that the composite mapping $gf : X \to Z$ is strongly closed. Then the mapping g is also strongly closed.

Lemma 2. Let $f : X \to Y$ be a strongly closed mapping, and let X_0 be such a closed subset of the set X that $fX_0 = fX$. Then the mapping $f_0 : X_0 \to Y$ (where f_0 is the restriction of the mapping f) will be strongly closed.

Corollary. Let $f : X \to Y$ be a strongly closed mapping of the space X onto the space Y. Then there exists such a closed subspace $X_0 \subset X$ that the mapping $f_0 : X_0 \to Y$ will be an irreducible strongly closed mapping.

Lemma 3. Let $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ be strongly closed bicompact mappings. Then there exists a space X and perfect mappings $g_1 : X \to X_1$ and $g_2 : X \to X_2$ such that $f_1 g_1 = f_2 g_2$, and this composite mapping is strongly closed.

Lemma 4. Let $\{X_\alpha; f_\alpha^\beta \mid \alpha, \beta \in A\}$ be an inverse spectrum of topological spaces with perfect projections. Suppose, moreover, that the indexing set A has a least element 0, and all projections $f_0^\alpha : X_\alpha \to X_0$ are strongly closed. Then the mapping $f_0 : X \to X_0$, where X is the inverse limit of the spectrum and f_0 is the natural projection, is strongly closed.

Definition. A set $\{f_\alpha : X_\alpha \to X \mid \alpha \in A\}$ of mappings onto a given topological space X will be called **directed**,

* All spaces considered here are Hausdorff.

** By $f^{\#}A$ is denoted the small image of the set A ($f^{\#}A = \{y \mid f^{-1}(y) \subset A\}$) in the sense of V. I. Ponomarev.

if for any two elements α_1 and α_2 of the set A there exists an element $\alpha \in A$ such that the mapping f_α covers the mappings f_{α_1} and f_{α_2}, i.e. there exist mappings $g_{\alpha_i}^\alpha : X_\alpha \to X_{\alpha_i}$, $i = 1, 2$, such that $f_\alpha = f_{\alpha_i} g_{\alpha_i}^\alpha = f_{\alpha_2} g_{\alpha_2}^\alpha$.

The directed set of mappings $\{f_\alpha : X_\alpha \to X \mid \alpha \in A\}$ can be completed by all possible projections g_β^α to an inverse spectrum, if the set A is ordered as follows: $\alpha_1 \geq \alpha_2$ if the mapping f_{α_1} covers the mapping f_{α_2}. We shall call this inverse spectrum the **spectrum generated by the set** $\{f_\alpha : X_\alpha \to X \mid \alpha \in A\}$.

The directed set of mappings $\{f_\alpha : X_\alpha \to X \mid \alpha \in A\}$ will be called **complete** if there exists an index $\alpha_0 \in A$ such that the limit X_A of the inverse spectrum generated by the set $\{f_\alpha : X_\alpha \to X \mid \alpha \in A\}$ is homeomorphic to the space X_{α_0}, and the limit projection $f_A : X_A \to X$ coincides with the mapping $f_{\alpha_0} : X_{\alpha_0} \to X$.

Lemma 5. *Let $\{f_\alpha : X_\alpha \to X \mid \alpha \in A\}$ be a complete set of mappings onto the space X. Then the space X has a maximal preimage with respect to this set of mappings.*

From Lemmas 2, 3, 4, 5 it follows:

Theorem 1. *For every Hausdorff space X there exist a space $\dot{X}$ and a bicompact irreducible strongly closed mapping $\pi : \dot{X} \to X$ such that the mapping π covers every bicompact irreducible strongly closed mapping $f : Y \to X$.*

Remark. In contrast to closed mappings, the composition of strongly closed mappings need not be strongly closed. Therefore, as a rule, there exist strongly closed irreducible bicompact mappings onto the space $\dot{X}$ that are not homeomorphisms.

Let now $f : X \to Y$ be a strongly closed mapping and M an arbitrary subset of the space Y. Consider the space Y^M, which is the quotient image of the space X under the mapping $f^M : X \to Y^M$, defined as follows:

$$f^M(x_1) = f^M(x_2) \Longleftrightarrow f(x_1) = f(x_2) \in Y \setminus M.$$

Then the space Y^M is mapped onto Y by means of the natural projection $\pi^M : Y^M \to Y$. Moreover, the mappings f^M and π^M are continuous and $f = \pi^M f^M$. It follows from Lemma 1 that the mapping π^M is strongly closed.

Lemma 6. *The space Y^M, constructed above, is Hausdorff.*

§ 2. In this paragraph a construction will be given of all bicompact irreducible strongly closed mappings onto an arbitrary normal space X. First we consider the case where X is bicompact.

Theorem 2. *In order that a continuous mapping $f : Y \to X$ onto a bicompactum be an irreducible bicompact strongly closed mapping, it is necessary and sufficient that for every point $x \in X$ the space $X^{\{x\}}$ be a bicompact extension of the space $X \setminus \{x\}$.*

Corollary 1. *The partially ordered set of irreducible bicompact strongly closed mappings onto a bicompactum X is isomorphic to the partially ordered set $\prod_{x \in X} C_x$, where C_x is the naturally ordered set of all bicompact extensions of the space $X \setminus \{x\}$.*

Corollary 2. *The preimage $\pi^{-1}(x)$ of each point x of the bicompactum X is homeomorphic to the remainder of the Stone–Čech extension of the space $X \setminus \{x\}$.*

Strongly closed bicompact irreducible mappings onto arbitrary completely regular spaces do not admit such a good description. There exists an example of a strongly closed bicompact irreducible mapping of a non-completely regular space onto a completely regular one ([1]). However, the following holds:

Theorem 3. Let $f : X \to Y$ be a strongly closed bicompact irreducible mapping of a Hausdorff space X onto a normal space Y. Then X will be a completely regular space.

Now strongly closed bicompact irreducible mappings onto normal spaces are described as follows.

Theorem 4. In order that a continuous mapping $f : X \to Y$ onto a normal space be a strongly closed bicompact irreducible mapping, it is necessary and sufficient that there exist a strongly closed irreducible mapping $\tilde{f} : bX \to \beta Y$ of some bicompact extension of the space X onto the Stone–Čech extension of the space Y, such that $\tilde{f}|_X = f$.

There are corollaries analogous to the corollaries of Theorem 2.

Mechanics and Mathematics Faculty
of M. V. Lomonosov Moscow State University

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Note: Figure translations are in progress. See original paper for figures.