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Abstract

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ON THE REPRESENTATION OF EXPANSIONS IN NORMAL WAVES BY CONTOUR INTEGRALS

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1. The boundary-value problem determining the stationary field $E(M)e^{-i\omega t}$, $H(M)e^{-i\omega t}$, excited by currents $j(M)e^{-i\omega t}$ in a layered medium characterized by tensors of dielectric and magnetic permeability $\varepsilon(\zeta)=\varepsilon'(\zeta)+i\varepsilon''(\zeta)$ and $\mu(\zeta)=\mu'(\zeta)+i\mu''(\zeta)$ (M is a point of the medium with coordinates ξ,η,ζ; $\zeta=$ const is the surface of a layer), can be written in the form

$$(L_{\xi,\eta}+L_\zeta)Z=F, \qquad Z\in\mathcal{H}, \tag{1}$$

where $L_{\xi,\eta}$ and L_ζ are linear differential operators in the coordinates ξ,η and ζ, with coefficients depending on ξ,η, and ζ, determined by Maxwell' s equations and the boundary conditions. They act on the desired function $Z(M)$, which determines $E(M)$ and $H(M)$. $F(M)$ is the prescribed function of the external action, determined by the currents $j(M)$. $L_{\xi,\eta}$ and L_ζ are defined on the sets $D(L_{\xi,\eta})$ and $D(L_\zeta)$, dense in the Hilbert spaces $H_{\xi,\eta}$ and H_ζ, $H_{\xi,\eta}\times H_\zeta=\mathcal{H}$. $L_{\xi,\eta}$ is a self-adjoint operator; it depends on the configuration of the layers and the choice of ξ and η, while the dissipative operator L_ζ depends on the functions $\varepsilon(\zeta)$ and $\mu(\zeta)$. The solution of (1) can be represented in the form of an expansion in normal waves propagating along the layers, i.e. in the coordinates ξ and η. There are two methods for obtaining it. The first gives this expansion *deus ex machina* from a certain contour integral evaluated by the method of residues ([1,2]).* The second method consists in expanding the solution of (1) in the eigenfunctions of L_ζ and source-wise in the coordinates ξ and η ([3–6]); in ([3–6]) we called it the method of normal waves. Below the connection between these methods will be established, and it will be shown that both approaches are based on the spectral theory of singular non-self-adjoint operators applied to expansions in eigenfunctions in a generalized sense ([7]).

2. The **expansion in normal waves propagating along the layers**, in its most general form, can be obtained from the contour integral

$$Z(\xi,\eta,\zeta) = -\frac{1}{2\pi i}\int_{K_\lambda}(L_\zeta - \lambda E)^{-1}(L_{\xi,\eta} + \lambda E)^{-1}F\,d\lambda, \tag{2}$$

where K_λ is a contour enclosing all singularities of the resolvent $(L_\zeta - \lambda E)^{-1}$ of the operator L_ζ and containing no singularities of the resolvent $(L_{\xi,\eta} - \mu E)^{-1}$ of the operator $L_{\xi,\eta}$ $(\mu = -\lambda)$. Representation (2) is a generalization of the Cauchy method (^{8}) (see also (^{9,10})), which consists in representing the expansion in the eigenfunctions of an operator $\mathcal{L}$ in the form of a contour integral of its resolvent $(\mathcal{L} - \lambda E)^{-1}$. In expanded form (2) will be

$$Z(\xi,\eta,\zeta) = -\frac{1}{2\pi i}\int_{K_\lambda} d\lambda \int G(\zeta,\zeta';\lambda)\iint G(\xi,\eta,\xi',\eta';-\lambda)F(\xi',\eta',\zeta')\,d\xi'\,d\eta'\,d\zeta', \tag{2'}$$

where $G(\zeta,\zeta';\lambda)$ is the kernel (i.e. the Green' s function) of the resolvent of L_ζ, and $G(\xi,\eta,\xi',\eta';\mu)$ is the kernel of the resolvent of $L_{\xi,\eta}$. If the spectrum of L_ζ consists only of nonmultiple–

* Therefore these waves are sometimes called residue waves (residue waves) (^{12}).

of the eigenvalues λ_k, determined from the equation

$$L_\zeta X(\zeta) = \lambda X(\zeta), \tag{3}$$

then $G(\zeta,\zeta';\lambda)$ will be a meromorphic function in the plane $\lambda = \lambda' + i\lambda''$,

$$G(\zeta,\zeta';\lambda) = H_\zeta(\zeta,\zeta';\lambda)/\Delta_\zeta(\lambda), \qquad \Delta_\zeta = \prod_{k=1,2,\ldots}(\lambda - \lambda_k), \tag{4}$$

where H_ζ is an entire function. Substituting (4) into (2′) and taking residues at the poles $\lambda = \lambda_k$, we obtain, in the case of a point source $F = \delta(M, M_0)$ (δ is the delta function), the expansion in normal waves

$$Z(\xi,\eta,\zeta) = \sum_{k=1}^{\infty}\frac{H_\zeta(\zeta,\zeta_0;\lambda_k)}{\frac{d}{d\lambda}\Delta_\lambda\Big|_{\lambda=\lambda_k}}G(\xi,\eta,\xi_0,\eta_0,-\lambda_k). \tag{5}$$

The phase factor of these waves is determined from the equation

$$L_{\xi\eta}G + \lambda_{kG} = \delta(\xi,\eta,\xi_0,\eta_0). \tag{6}$$

Here it is assumed that $\mu = -\lambda_k$ are not irregular points of the resolvent $L_{\xi\eta}$. The question of convergence of (5) is resolved by constructing an infinite sequence of contours $\{K_\lambda\}$. If it is known that such an expansion exists, then, according to (3–6), one may seek the solution (5), substituting into (1) the expansion $Z = \sum Y_k(\xi,\eta) X^k(\zeta)$, where $X^k(\zeta)$ are eigenfunctions of L_ζ satisfying (3). Taking into account the biorthogonality condition $(X^k, U^p) = N_k \delta_{kp}$, where U^p are eigenfunctions of the adjoint operator L_ζ^*, and N_k is a normalizing factor, we obtain (5) in the form

$$Z(\xi,\eta,\zeta) = \sum_{k=1}^{\infty} \frac{X^k(\zeta)\overline{U}^k(\zeta_0)}{N_k} G(\xi,\eta,\xi_0,\eta_0;-\lambda). \tag{7}$$

Comparing (5) and (7), we obtain the important relation

$$N_k = \frac{X^k(\zeta)\overline{U}^k(\zeta_0)}{H_\zeta(\zeta,\zeta_0;\lambda_k)} \frac{d}{d\lambda}\Delta_\zeta\Big|_{\lambda=\lambda_k}. \tag{8}$$

In the case of an arbitrary $F(M)$, the expansion in normal waves will be

$$Z(\xi,\eta,\zeta) = \sum_{k=1}^{\infty} \frac{X^k(\zeta)}{N_k} \iint G(\xi,\eta,\xi',\eta';-\lambda_k)(F,U^k)\,d\xi'\,d\eta'. \tag{9}$$

If the spectrum of L_ζ is discrete, but degeneracy occurs, there exists an expansion in normal and associated waves,

$$Z(\xi,\eta,\zeta) = \sum_{s=1}^{\infty}\sum_{r=1}^{m_s} X_s^r(\zeta) \sum_{k=1}^{r} \frac{N_k}{N_r} \prod_{1,2,\ldots,(r-k+1)} \left(L_{\xi\eta} + \lambda_{sE}\right)^{-1} F_s^k, \tag{10}$$

where $\prod$ is the product of $(r-k+1)$ resolvents of the operator $L_{\xi\eta}$.

Associated waves differ from normal waves in that their amplitudes grow with distance from the source as polynomials of degree $(r-1)$. They are forced waves in the regime of wave resonance (the wave number of the wave of the external action, i.e. the normal wave, coincides with the wave number of the associated wave). In the general case, the spectrum of the operator L_ζ consists of points λ at which $(L_\zeta - \lambda E)^{-1}$ either does not exist, or is unbounded, or is not defined everywhere in $\hat{H}_\zeta$, and the computation (2) gives an expansion in the discrete and continuous spectra of normal and associated waves.

3. **The expansion in normal waves traveling across the layers** can be obtained in an analogous way if one interchanges L_ζ and

$L_{\xi\eta}$ in (2). Then we obtain the contour integral

$$Z(\xi,\eta,\zeta) = -\frac{1}{2\pi i}\int_{K_\mu} (L_{\xi\eta} - \mu E)^{-1}(L_\zeta + \mu E)^{-1} F\,d\mu, \tag{11}$$

where the contour K_μ encloses all singularities of the resolvent $L_{\xi\eta}$ and contains no singularities of the resolvent L_ζ. If the spectrum of $L_{\xi\eta}$ is discrete and simple, i.e., the only irregular points of the resolvent $L_{\xi\eta}$ are the eigenvalues μ_l $(l = 1, 2, ...)$, at which $(L_{\xi\eta} - \mu E)^{-1}$ does not exist, determined from the equation

$$L_{\xi\eta} Y^l(\xi, \eta) = \mu^l_{lY}(\xi, \eta), \tag{12}$$

then $G(\xi, \eta, \zeta', \eta', \mu)$ will be a meromorphic function. Taking the residues of the integral (11), we obtain the expansion

$$Z(\xi, \eta, \zeta) = \sum_{l=1}^{\infty} \frac{Y^l(\xi, \eta)}{M_l} \int G(\zeta, \zeta'; -\mu_l)(F, V^l)\, d\zeta, \tag{13}$$

where $M_l = (Y^l, V^l)$ is a normalizing factor; Y^l and V^l are eigenfunctions of the operator $L_{\xi\eta}$ and of the adjoint operator $L^*_{\xi\eta}$. The Green' s function $G(\zeta, \zeta'; -\mu_l)$ is determined from the equation $L_\zeta G + \mu_l G = \delta(\zeta, \zeta')$. In the case $F = \delta(M, M_0)$, from (13) we obtain

$$Z(\xi, \eta, \zeta) = \sum_{l=1}^{\infty} \frac{Y^l(\xi, \eta)\bar{V}^l(\xi_0, \eta_0)}{M_l} G(\zeta, \zeta'; -\mu_l). \tag{13'}$$

Each term of $(13')$ represents a normal wave propagating in the coordinate ζ, i.e., across the layers, while preserving the form $Y^l(\xi, \eta)$.

4. The transition from expansions in normal waves propagating along the layers to expansions in waves propagating across the layers is effected by replacing λ by $-\mu$ and by the corresponding deformation of the contour K_λ into K_μ in the integral (2), as a result of which it turns into the integral (11). This remarkable transformation was first used by Sommerfeld in (1) for the special case of a plane-layered isotropic medium, and by Watson in (2) for two special cases of a spherically layered isotropic medium, for which expansions in spectra of normal waves propagating across the layers were well known. Having constructed (11) for them and transformed it to (2), they reduced the problem of expansion in normal waves running along the layers to contour integration (2) by the method of residues. Our line of investigation ($^{3-6}$) consists in the direct construction of (5), or (9), or (10) by methods of the spectral theory of singular non-self-adjoint operators, which has been successfully developed in recent decades (^{11}). In the case when the spectra of L_ζ and $L_{\xi\eta}$ are discrete and simple, the transition from (13) to (9) can be made without using contour integrals. For this purpose we expand the Green' s functions L_ζ and $L_{\xi\eta}$ in the corresponding spectra:

Fig. 1. Diagram of relationships among spectral, source-type, and normal-wave expansions.

Figure 1: Fig. 1. Diagram of relationships among spectral, source-type, and normal-wave expansions.

$$G(\zeta,\zeta';\lambda)=\sum_{k=1}^{\infty}\frac{X^k(\zeta)\bar{U}^k(\zeta')}{(\lambda_k-\lambda)N_k},\qquad G(\xi,\eta,\xi',\eta';\mu)=\sum_{l=1}^{\infty}\frac{Y^l(\xi,\eta)\bar{V}^l(\xi',\eta')}{(\mu_l-\mu)M_l}. \tag{14}$$

Substituting the first of expressions (14) into (13), we obtain a purely spectral representation of the solution (1)

$$Z(\xi,\eta,\zeta)=\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\frac{X^k(\zeta)Y^l(\xi,\eta)}{M_{kM}l(\lambda_k+\mu_l)}\iiint F\bar{U}^k\bar{V}^l\,d\xi'\,d\eta'\,d\zeta'. \tag{15}$$

This is an expansion in normal oscillations. By changing in it the order of summation and contracting the sum with respect to the index l into the Green' s function, according to the second expression (14), we obtain the expansion (9). The inverse transition is analogous. Figure 1 presents the connections between the various methods used in solving (1). Let us note that the diffraction-ray method used in (^{12,13}) consists in replacing the subintegral expression

in (2) or (11) by its asymptotic approximation for large $\lambda(\mu)$ and calculation of the integral by the saddle-point method.

5. The expansion in normal waves (5) differs from expansions in the spectrum in eigenfunctions of the operator L_ξ by the summing factor $G(\xi,\eta,\xi_0,\eta_0;-\lambda_k)$, which improves convergence. Therefore the requirements on the operator L_ξ that ensure convergence of (5) are considerably weaker than those for convergence of expansions in the spectrum of L_ξ. This is a very important fact, since in most practically important cases the system of functions,

Fig. 1

although complete in $\{X^k\}$, as a rule does not form a basis in it. In this case, for the expansion (5) it is only necessary that the series (5) converge for ξ,η not equal to ξ_0,η_0, and that as $\xi\to\xi_0$ and $\eta\to\eta_0$, (5) represent the expansion in the spectrum of L_ξ in the generalized sense (7). Thus, for each type of layer (plane, cylindrical, spherical, etc.) and for the chosen coordinate system ξ,η, there is a certain class of operators L_ξ for which an expansion in normal waves (5) exists. The investigation of classes of such operators has only just begun. Thus, for example, in (^{14}) a class of operators with discrete spectrum was studied that provides an expansion in normal waves in plane-layered media, when ξ and η are Cartesian coordinates.

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