ON THE INVARIANCE OF THE DIRAC EQUATIONS

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Abstract

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ON THE INVARIANCE OF THE DIRAC EQUATIONS

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This paper studies the question of the invariance of the Dirac equations with respect to a continuous group of transformations of the Pauli-transformation type. The broadest groups of transformations of this type, admissible in the sense of S. Lie by the Dirac equations for nonzero and zero masses, are found. In the case of zero mass, the group found has 7 essential parameters and contains, as a subgroup, the known 4-parameter Pauli group. A 6-parameter subgroup of this group, isomorphic to the group of 4-dimensional rotations, is singled out. In the case of mass different from zero, the group found is a 3-parameter group isomorphic to the group of three-dimensional rotations.

Pauli discovered ([1]) a three-parameter continuous group of transformations with respect to which the Dirac equations

$$(\gamma_\mu^T \partial/\partial x^\mu - m)\psi = 0 \tag{2}$$

are invariant in the case $m = 0$, considering the transition to the complex conjugate of the function ψ. In this case, along with equations (1), one must also take into account the conjugate equations, which can be written in the form

$$(\gamma_\mu^T \partial/\partial x^\mu - m)\overline{\psi} = 0$$

with the function

$$\overline{\psi} = \gamma_4 \psi^*. \tag{3}$$

The asterisk denotes complex conjugation, and T denotes the transposition operation. This group, together with the group of transformations

$$\psi' = (\cos\alpha + i\sin\alpha \cdot \gamma_5)\psi,$$

$$\overline{\psi}' = (\cos\alpha + i\sin\alpha\cdot\gamma_5)\overline{\psi}, \tag{4}$$

where $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, α is a real parameter, is called the 4-parameter Pauli group. The structure of the Pauli group was investigated in ([2]).

We shall show that the Pauli group is not the broadest continuous group of transformations of the form

$$\begin{pmatrix}\psi' \\ \overline{\psi}'\end{pmatrix} = A\begin{pmatrix}\psi \\ \overline{\psi}\end{pmatrix}, \tag{5}$$

where $\begin{pmatrix}\psi \\ \overline{\psi}\end{pmatrix}$ is an 8-component column vector, and A is a complex square matrix of order 8 (generally speaking, depending on x), admissible in the sense of S. Lie by equations (1), (2), and we shall find the broadest group both in the case $m = 0$ and for arbitrary mass m. We note that consideration of the pair of complex equations (1) and (2) is equivalent to separating equation (1) into real and imaginary parts. In the latter case it is natural to consider groups of real transformations. The reality condition for the transformations in this case leads to the equality

$$\overline{\psi}' = \overline{\psi'}. \tag{6}$$

From (5) and (6) it follows that the matrix A has the form (γ_4 is real)

$$A = \begin{pmatrix} a & b \\ \gamma_4 b^*\gamma_4 & \gamma_4 a^*\gamma_4\end{pmatrix} \tag{7}$$

with arbitrary fourth-order matrices a and b. The equivalence of conditions (6) and (7) is verified without difficulty.

To write down explicitly the admissible group, let us choose a definite representation of the matrices γ_μ ($\mu = 1, \dots, 4$):

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 1\end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0\end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0\end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1\end{pmatrix}. \tag{8}$$

We shall need the following properties of the matrices (8):

$$\gamma_1^T = -\gamma_1, \quad \gamma_2^T = \gamma_2, \quad \gamma_3^T = -\gamma_3, \quad \gamma_4^T = \gamma_4, \quad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}. \tag{9}$$

By the well-known method of the theory of group properties of differential equations (3), we can find the broadest admissible groups. We formulate the results obtained as two theorems, respectively for the cases $m = 0$ and $m \neq 0$.

Theorem 1. *The broadest continuous group of transformations (5) satisfying condition (6), admitted in the sense of S. Lie by equations (1), (2) for $m = 0$, contains 7 essential parameters. It consists of the 6-parameter group G_6, isomorphic to the group of four-dimensional rotations O_4, and the group of transformations (4). The 3-parameter Pauli group is a subgroup of the group G_6.*

We shall write here the basis infinitesimal operators (generators) of the group G_6 and establish an isomorphism between G_6 and O_4. With the aid of the matrices (8), the basis operators of the group G_6 can be written in the form of the following square eighth-order matrices:

$$S_1 = \begin{pmatrix} 0 & \gamma_4\gamma_2 \\ -\gamma_4\gamma_2 & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & i\gamma_4\gamma_2 \\ i\gamma_4\gamma_2 & 0 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} 0 & \gamma_3\gamma_1 \\ \gamma_3\gamma_1 & 0 \end{pmatrix}.$$

$$S_4 = \begin{pmatrix} 0 & i\gamma_3\gamma_1 \\ -i\gamma_3\gamma_1 & 0 \end{pmatrix}, \qquad S_5 = \begin{pmatrix} -\gamma_5 & 0 \\ 0 & \gamma_5 \end{pmatrix}, \qquad S_6 = \begin{pmatrix} -i1 & 0 \\ 0 & i1 \end{pmatrix}. \tag{10}$$

Here 0 and 1 denote, respectively, the zero and unit matrices. The transformations of the group G_6 are easily written down, taking into account the equalities

$$S_1^2 = S_2^2 = S_5^2 = 1, \qquad S_3^2 = S_4^2 = S_6^2 = -1, \tag{11}$$

by the formula*

$$\begin{pmatrix} \psi' \\ \bar{\psi}' \end{pmatrix} = e^{\alpha S} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \quad (\alpha \text{ is a real parameter}). \tag{12}$$

Let us write down, as an example, the one-parameter transformation groups for the operators S_1 and S_3. We have, taking (11) into account,

$$e^{\alpha S_1} = 1 + \alpha S_1 + \frac{\alpha^2}{2!} S_1^2 + \cdots = \operatorname{ch}\alpha + S_1 \operatorname{sh}\alpha.$$

Therefore formula (12) gives the one-parameter group

$$\psi' = \operatorname{ch}\alpha\cdot\psi + \operatorname{sh}\alpha\cdot\gamma_4\gamma_2\bar{\psi}, \qquad \bar{\psi}' = \operatorname{ch}\alpha\cdot\bar{\psi} - \operatorname{sh}\alpha\cdot\gamma_4\gamma_2\psi. \tag{13}$$

* The connection between the writing of infinitesimal operators in Lie form and the matrix writing (10) is indicated in ([4]). Formula (12) is equivalent to finding the transformations of the group by solving the Lie equations.

For S_3 we have, analogously,

$$e^{\alpha S_3} = \cos\alpha + S_3\sin\alpha, \qquad \psi' = \cos\alpha\cdot\psi + \sin\alpha\cdot\gamma_3\gamma_1\bar{\psi},$$
$$\bar{\psi}' = \cos\alpha\cdot\bar{\psi} + \sin\alpha\cdot\gamma_3\gamma_1\psi. \tag{14}$$

Comparing the transformations of the group G_6 with the Pauli group, one can establish the following. The operators S_3, S_4, and S_6 are basis operators of the 3-parameter Pauli group in the representation (8). The one-parameter groups of the operators S_1, S_2, S_5 do not enter the Pauli group. With respect to the operators S_1 and S_2 this fact can be established from other considerations (this is done in Remark 2).

Let us construct an isomorphism of the groups G_6 and O_4. Introduce the notation

$$X_{12} = \frac{i}{2}S_2, \qquad X_{23} = \frac{1}{2}S_4, \qquad X_{31} = -\frac{i}{2}S_5, \qquad X_{14} = \frac{i}{2}S_1,$$
$$X_{24} = -\frac{1}{2}S_6, \qquad X_{34} = -\frac{1}{2}S_3. \tag{15}$$

It follows from (4) that the commutators of infinitesimal operators in Lie form correspond to the "commutators" of the matrices (10), defined as

$$[S_i, S_j] = S_{jS}i - S_{iS}j. \tag{16}$$

Computing the commutators of the operators (15) by the formulas (16), we arrive at the structure of the group O_4:

$$[X_{\mu\nu}, X_{\sigma\tau}] = \delta_{\mu\sigma}X_{\nu\tau} + \delta_{\nu\tau}X_{\mu\sigma} - \delta_{\mu\tau}X_{\nu\sigma} - \delta_{\nu\sigma}X_{\mu\tau}. \tag{17}$$

The infinitesimal operator of the group (4) commutes with all the operators (10).

Theorem 2. *The broadest continuous group of transformations (5), under condition (6), admitted by equations (1), (2) for* $m \neq 0$*, has 3 parameters and is isomorphic to the group of three-dimensional rotations* O_3*. The basis operators of this group may be chosen as* S_1, S_2*, and* S_6*.*

As in Theorem 1, so also in this theorem, we have excluded from consideration the trivial dilation transformation of the functions ψ and $\bar{\psi}$.

The invariance of the Dirac equations (1), (2) with respect to the one-parameter groups of transformations indicated in Theorem 2 can easily be checked. Let us do this, for example, for the operator S_1. The one-parameter group of transformations for this operator has the form (13). Substituting (13) into (1), (2) and taking (9) into account, we obtain

$$(\gamma_\mu \partial/\partial x_\mu + m)\psi' = \operatorname{ch}\alpha(\gamma_\mu \partial/\partial x^\mu + m)\psi + \operatorname{sh}\alpha(\gamma_\mu \partial/\partial x^\mu + m)\gamma_4\gamma_2\bar{\psi} =$$
$$= \operatorname{sh}\alpha \cdot \gamma_4\gamma_2(\gamma_1\partial/\partial x^1 - \gamma_2\partial/\partial x^2 + \gamma_3\partial/\partial x^3 - \gamma_4\partial/\partial x^4 + m)\bar{\psi} =$$
$$= \operatorname{sh}\alpha \cdot \gamma_4\gamma_2(-\gamma_\mu^T\partial/\partial x^\mu + m)\bar{\psi} = 0$$

$$(\gamma_\mu^T \partial/\partial x^\mu - m)\bar{\psi}' = \operatorname{ch}\alpha(\gamma_\mu^T \partial/\partial x^\mu - m)\bar{\psi} + \operatorname{sh}\alpha(\gamma_\mu^T \partial/\partial x^\mu - m)\gamma_4\gamma_2\psi =$$
$$= \operatorname{sh}\alpha\, \gamma_4\gamma_2(\gamma_\mu\partial/\partial x^\mu + m)\psi = 0,$$

which is precisely the required invariance of the Dirac equations.

Remark 1. From the invariance of the Dirac equations (1), (2) with respect to any one-parameter group with an operator that is a linear combination of the operators S_3, S_4, S_5, and also with respect to the group (4), there follows the condition $m = 0$.

Remark 2. No linear combination of the operators S_1 and S_2 can belong to the operators of the Pauli group. Indeed, if this were possible even for one operator obtained in this way, we would obtain a one-parameter group belonging to the Pauli group, with respect to which the Dirac equations with nonzero mass are invariant, which is impossible ([2], p. 368).

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Note: Figure translations are in progress. See original paper for figures.