ON THE ASYMPTOTICS OF THE WAVE FUNCTION OF A QUASISTATIONARY STATE

1969

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Fig. 1

Figure 1: Fig. 1

Abstract

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UDC 517.944/947

MATHEMATICAL PHYSICS

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ON THE ASYMPTOTICS OF THE WAVE FUNCTION OF A QUASISTATIONARY STATE

(Presented by Academician A. N. Tikhonov on 26 II 1969)

I. Let R_N be an N-dimensional Euclidean space, $N \geq 3$, $x = (x_1, x_2, \dots, x_N)$, Δ the Laplace operator, $V(x)$ a scalar function (potential), and H the operator defined on finite functions by the formula

$$Hu = -\Delta u + V(x)u. \quad (1)$$

Suppose that in R_{N+1} the section of the surface $z = V(x)$ by a plane passing through the z-axis has the form shown in Fig. 1; let $\psi(x,t)$ be the solution of the Cauchy problem for the Schrödinger equation

$$\partial\psi/\partial t = -iH\psi, \qquad \psi(x,+0) = \psi_0(x);$$

the support of the function $\psi_0(x)$ is contained in the ball $|x| < R$, and

$$E_0 = \langle \psi_0, H\psi_0 \rangle \ll V_0. \quad (2)$$

Fig. 1

From physical considerations ([1,2]) it follows that, for a special choice of $\psi_0(x)$ and under condition (2),

$$\psi(x,t) \sim \exp(-i\lambda t)A(x) + B(x)t^{-3/2},$$

$$t \to +\infty,$$

where the coefficient $B(x)$ is negligibly small, $\operatorname{Re}\lambda$ is the quasilevel energy, and $\operatorname{Im}\lambda$ is its width. Our article is devoted to a rigorous proof of this fact. The main result is formulated by us in Theorem 1.

II. Let $V(x)$ be a nonnegative function, $\Omega = \{x,\ V(x) = \infty\}$, Ω_1 a connected component of the set $R_N \setminus \Omega$ containing infinitely distant points, and $\Omega_2 = R_N \setminus (\Omega \cup \Omega_1)$. We assume that $V(x)$ satisfies the conditions $A(R)$, which consist in the following: 1) for any $M > 0$ the function $V_M(x) = \min\{V(x), M\}$ locally satisfies the Hölder condition; 2) there exists such an $R < \infty$ that $V(x) = 0$ for $|x| > R$, and we assume that $\operatorname{mes}\Omega_2 > 0$.

By the symbol H we shall denote the extension of the operator (1) constructed in § 2 of ([3]), and by the symbol H_M the analogous extension of the operator

$$H_{Mu} = -\Delta u + V_M(x)u;$$

the remaining notation is borrowed by us from ([3–5]), and

$$\theta(M) = \left(\iint |g_M(x,y,\tau) - g(x,y,\tau)|\, dx\, dy\right)^{1/2}.$$

III. Suppose that the operator H has a nontrivial discrete spectrum $\{\lambda_j\}$ and that $\{\psi(x,\lambda_j)\}$ are the eigenfunctions of the discrete spectrum of the operator H.

Definition. The solution of the abstract Cauchy problem

$$\partial\psi/\partial t = -iH_M\psi; \qquad \psi \subset L^2; \qquad \psi(x,+0) = \psi(x,\lambda_j) \tag{3}$$

we call the wave function of the quasistationary state and denote by the symbol $\psi_M(x,t,\lambda_j)$.

The solvability of problem (3) is proved by us on the basis of the ideas of work ([6]). In the physical formulation, the problem we solve is formulated as follows. Suppose we have an infinite potential barrier (i.e. $V_0 = \infty$), and suppose that inside it there is a particle in a stationary state. (This means that the wave function of the particle is $\exp(-i\lambda_j t)\psi(x,\lambda_j)$.) At the time $t = +0$ we have changed the potential barrier from infinite to finite, but sufficiently large ($\lambda_j \ll V_0$). (The inequality $\lambda_j/d > 1$ is not assumed.) The initial state of the particle ceases to be stationary. We compute, to first order in the small* parameter $\theta(V_0)$, the asymptotics of the wave function of the particle as $t \to +\infty$.

IV. **Lemma 1.** *If the potential $V(x)$ satisfies the conditions $A(R)$, λ_j is a simple eigenvalue of the discrete spectrum of the operator H, and $M > M_j$, then the function $R(1, T_M^+(\lambda)) = (E - T_M^+(\lambda))^{-1}$, considered as an element of the space $[\mathfrak{A} \to \mathfrak{A}]$, where $\mathfrak{A}$ is the Hilbert space with scalar product*

$$\{f_1, f_2\} = \int f_1^*(x) f_2(x) \exp(-|x|)\, dx$$

is holomorphic in λ *at all points of some disk* $\{\lambda;\ |\lambda - \lambda_j| < \varepsilon\}$*, except for the single point* $\lambda_j^+(M)$*,* $\operatorname{Im} \lambda_j^+(M) < 0$*, and* $\lambda_j^+(M)$ *is a pole of first order.*

Theorem 1. *If the conditions of Lemma 1 are fulfilled and* $M > M_j$*, then the function* $\psi_M(x, t, \lambda_j)$ *can be represented as the sum of three functions*

$$\psi_M(x, t, \lambda_j) = \exp(-i\lambda_j^+(M)t) A_j(x, M) + B_j(x, M, t) + D_j(x, M, t),$$

where

$$\begin{aligned} A_j(x, M) = &-(\pi i)[\lambda_j^+(M)]^{N/2-1}(2\pi)^{-N} \int_{|n|=1} dn\ \operatorname{Res}\Big\{ (R(1, T_M^-(\lambda))\times \\ &\times T_M^-(\lambda) \exp(-i\sqrt{\lambda}(n, y)))(x) \int [\exp(i\sqrt{\lambda}(n, y)) + R(1, T_M^+(\lambda))\times \\ &\times \exp(i\sqrt{\lambda}(n, y))] \psi(y, \lambda_j)\, dy, \quad \lambda = \lambda_j^+(M) \Big\}, \end{aligned}$$

the functions $B_j(x, M, t)$ *and* $D_j(x, M, t)$ *belong to* L^2 *and are such that*

$$\|B_j\|_\infty + \|D_j\|_2 < C_j \theta(M)^{1/5},$$

the constant C_j *depends only on* λ_j*.*

It can be shown that $A_j \in L^2$ and

$$A_j(x, M) = O(|x|^{-N-2}), \qquad |x| \to \infty,$$

and the point $\lambda_j^+(M)$ is a pole of the analytic continuation of the scattering amplitude $f_M(\sqrt{\lambda}, n_{in}, n_{out})$ into the lower half-plane.

The number $\lambda_j^+(M)$ is approximately computed by the formula

$$\lambda_j^+(M) = \lambda_j + \operatorname{Sp}(P(T^+ - T_M^+)P) / \operatorname{Sp}\left(P \frac{\partial T^+}{\partial \lambda} P\right)\Bigg|_{\lambda=\lambda_j} + O(\theta(M)),$$

where

$$P = (2\pi i)^{-1} \int_{|1-\mu|=\delta} R(\mu, T^+(\lambda_j))\, d\mu,$$

δ is sufficiently small, and $\tau \approx \rho(M, M/2)M^{-1/2}$.**

V. Earlier ([2,7]) an analogous problem was solved in a somewhat different formulation, and the wave function of the quasistationary state obtained was exponentially increasing in modulus as $|x| \to \infty$.

* Under very broad assumptions $\theta(M) \sim \exp(-\beta M^{1/2-\delta})$, $\beta > 0$, $\delta > 0$.

** Here τ is the quantity which in ([3–5]) we denoted by the symbol t.

But, according to the axioms of quantum mechanics, a function that is not square-integrable cannot be the wave function of a particle localized at the initial moment; therefore the interpretation of the solution obtained is not clear.

VI. The author expresses his gratitude to A. N. Tikhonov and V. A. Il' in for valuable consultations.

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Received
25 II 1969

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Note: Figure translations are in progress. See original paper for figures.