ON ITERATIVE METHODS FOR SOLVING SECTORAL PROBLEMS CONTAINING DISCRETE VARIABLES

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Abstract

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MATHEMATICS

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ON ITERATIVE METHODS FOR SOLVING SECTORAL PROBLEMS CONTAINING DISCRETE VARIABLES

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1°. The method described below for solving integer linear programming problems is based on ideas from iterative methods in game theory and convex programming (see [1, 2]) and on stochastic methods of integer programming proposed in [3, 4].

As is known, many problems of optimal planning for the development of a branch of production reduce to an integer programming problem of the following form:

Under the conditions

$$\text{I.}\quad \sum_{j=1}^{n}\sum_{k=1}^{r_j} a_{kj}^{i} z_{kj} \geq b_i, \qquad i = 1, \ldots, m;$$

$$\text{II.}\quad \sum_{k=1}^{r_j} z_{kj} \leq 1, \qquad j = 1, \ldots, n;$$

III. z_{kj} take the values 0 or 1,

minimize

$$\sum_{j=1}^{n}\sum_{k=1}^{r_j} c_{kj} z_{kj}.$$

In what follows, the integer linear programming problem with conditions I, II, III will be called **problem A**. Replacing the integrality condition on the variables z_{kj} by the conditions $z_{kj} \geq 0$, we obtain the corresponding **linear programming problem A***, whose solution may serve as an estimate for the integer problem.

2°. We describe an iterative method for solving problem A*, which consists in minimizing

$$\sum_{j=1}^{n}\sum_{k=1}^{r_j} c_{kj}x_{kj}$$

under the conditions:

$$1)\ \sum_{j=1}^{n}\sum_{k=1}^{r_j} a_{kj}^i x_{kj} \geq b_i, \qquad i = 1, \dots, m;$$

$$2)\ \sum_{k=1}^{r_j} x_{kj} \leq 1, \qquad j = 1, \dots, n;$$

$$3)\ x_{kj} \geq 0.$$

The dual **problem B** consists in maximizing

$$\sum_{i=1}^{m} p_i b_i - \sum_{j=1}^{n} q_i$$

under the conditions:

$$1)\ \sum_{i=1}^{m} a_{kj}^i p_i - q_j \leq c_{kj};$$

$$2)\ p_i \geq 0 \text{ and } q_j \geq 0.$$

For solving problem $\mathbf{A}^*$, the following iterative process can be used (cf. (2)). Suppose that at the l-th iteration $x_{kj}^{(l)}$ and $p_i^{(l)}$ have been obtained. Put

$$q_j^{(l)} = \max_{1\leq k\leq r_j}\left\{\sum_{i=1}^{m} p_i^{(l)} a_{kj}^i - c_{kj},\ 0\right\},$$

$$\hat{x}_{kj}^{(l)} = \begin{cases} 0, & \text{if } \sum_{i=1}^{m} p_i^{(l)} a_{kj}^i - c_{kj} < q_j^{(l)}, \\ 1, & \text{if } \sum_{i=1}^{m} p_i^{(l)} a_{kj}^i - c_{kj} = q_j^{(l)}. \end{cases}$$

The iterative process is defined by the relations

$$x_{kj}^{(l+1)} = (1-\alpha_l)x_{kj}^{(l)} + \alpha_l \hat{x}_{kj}^{(l)}; \tag{1}$$

$$p_i^{(l+1)} = (1+\theta_i^{(l)} h_l) p_i^{(l)}, \tag{2}$$

where

$$\theta_i^{(l)} = \begin{cases} -1, & \text{if } \sum_{j=1}^{n}\sum_{k=1}^{r_j} a_{kj} x_{kj}^{(l+1)} \geq b_i \text{ and } \sum_{j=1}^{n}\sum_{k=1}^{r_j} a_{kj}^i \hat{x}_{kj}^{(l)} \geq b_i; \\ +1, & \text{if } \sum_{j=1}^{n}\sum_{k=1}^{r_j} a_{kj}^i x_{kj}^{(l+1)} < b_i \text{ and } \sum_{j=1}^{n}\sum_{k=1}^{r_j} a_{kj}^i \hat{x}_{kj}^{(l)} < b_i; \\ 0 & \text{in all other cases;} \end{cases}$$

the numbers α_l and h_l belong to the interval $[0,1]$ and tend monotonically to zero. For the iterative process to converge, the manner in which the sequences $\{\alpha_l\}$ and $\{h_l\}$ decrease is immaterial; it is only necessary that $\alpha_l \to 0$, $h_l \to 0$, and

$$\sum_{l=1}^{\infty} \alpha_l = \sum_{l=1}^{\infty} h_l = \infty$$

(see [1]). However, experiments show that the rate of convergence of the process depends strongly on the nature of the changes in α_l and h_l (see item 3°).

We note that the vectors $p^{(l)}, q^{(l)}$ at each iteration satisfy conditions 1) and 2) (a feasible plan of the dual problem), whereas the plans $x^{(l)}$ do not always satisfy conditions (1).

3°. The program implementing the algorithm for solving problem $\mathbf{A}^*$ provides for storing a solution x that is feasible (i.e., satisfies the conditions of problem $\mathbf{A}^*$) and corresponds to the best value of the criterion among all feasible solutions obtained over the course of previous iterations. By comparing this value with the best value of the dual problem obtained during the same time, one can judge how close the solution is to the optimum and, consequently, obtain solutions with prescribed accuracy.

We give one possible way of changing the parameters α_l and h_l and show, on a concrete example, how the rate of convergence depended on the nature of the changes in α_l and h_l.

Choosing a natural number d and initial values α_0 and h_0, we divide them simultaneously in half at the iterations with numbers $d, 2d, 4d, 8d, 16d$, etc. The experiment was carried out on a sectoral planning problem in which $n = 68$; $r_j = 2 \div 4$; $\sum_{j=1}^{n} r_j = 112$; $m = 7$ (see Table 1).

4°. We proceed to the description of the method for solving problem **A**. The difference between this method and the one considered in item 2° is that the variables $x_{kj}^{(l+1)}$ are used as probabilities for obtaining random integer-valued

Table 1

d	4	8	16	32	64	128
Criterion value	403,670	393,666	398,465	398,394	398,682	393,857
Number of iterations	215	83	199	688	267	850
Optimal criterion value				393,266		

variables $z_{kj}^{(l+1)}$, which satisfy the integrality condition (condition III) in problem A. Choosing for each j $(j = 1, \dots, n)$ a random number $\xi_j^{(l+1)}$ uniformly distributed on the interval $[0, 1]$, we set

$$z_{kj}^{(l+1)} = \begin{cases} 1, & \text{if } \sum_{s=1}^{k-1} x_{sj}^{(l+1)} \le \xi_j^{(l+1)} < \sum_{s=1}^{k} x_{sj}^{(l+1)}; \\ 0, & \text{otherwise.} \end{cases}$$

$p_i^{(l+1)}$ are determined by relations (2), where $x_{kj}^{(l+1)}$ are replaced by $z_{kj}^{(l+1)}$.

This iterative process is a Markov walk over variants of the solution, whose transition probabilities are formed in the course of the process itself. As is known, in a finite irreducible (ergodic) Markov chain the probability of reaching any state in a finite number of steps (iterations) is equal to one.

Table 2*

No.	L	m	N	V	C	T	x, %
1	112	8	68	398	399	2358	0,25
2	411	25	51	711	737	1219	5,2
3	73	174	31	314	333	1742	6,05
4	199	46	67	812	911	2401	12,2
5	199	26	63	388	423	1700	9,28
6	438	10	36	247	258	3398	4,4
7	438	10	36	146	149	11	2,1

* L is the total number of variables; m is the number of constraints; N is the number of enterprises; V is the optimal value of the criterion of the linear problem; C is the best value of the criterion of the problem with discrete variables obtained in the course of the computation; T is the iteration number at which the best solution was obtained; $x = (C - V)/V$.

By a small modification of the method for assigning the transition probabilities, the conditions for ergodicity of the chain can be observed. This guarantees the theoretical convergence of the method. However, because of the enormous number of states, the theoretical possibility of reaching the optimum in the enumeration process proves nothing, but indicates that in stochastic algorithms it is meaningless to speak of convergence without estimates of its rate.

5°. A series of problems was solved by the algorithm described. Each time before solving a problem containing discrete variables, the corresponding linear programming problem with continuous variables was solved. The optimal value of the criterion was used as an estimate of the solutions obtained for the main (i.e., discrete) problem.

After a number of computational experiments had been carried out, it became more or less clear that the rate of convergence (i.e., the time required to obtain a "good" solution) is weakly correlated with the initial values of the prices $p_i^{(0)}$ and the initial distribution $x_{jk}^{(0)}$. On the other hand, a strong dependence was observed on the rate of decrease of the damping parameters α_l and h_l. For the time being, however, it does not seem possible to find an "optimal" variation of the damping parameters.

Table 2 illustrates the solution of a number of problems on the BESM-6 computer.

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Note: Figure translations are in progress. See original paper for figures.