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Abstract

Full Text

MATHEMATICS

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ON METHODS, NONDETERMINISTIC AND OPTIMAL ON CLASSES OF FUNCTIONS, FOR SOLVING THE POISSON EQUATION

(Presented by Academician A. N. Tikhonov on 29 IV 1969)

1. Consider the first boundary-value problem for the Poisson equation $\Delta u = f$ in the unit square Ω, with homogeneous boundary condition and with right-hand side from a class of functions whose norm in the space of S. L. Sobolev W_q^r, $r \geqslant 1$, $q > 2$, does not exceed the constant M. We shall show that, using information about the right-hand side at $O(N^2)$ random nodes, one can find the solution with probabilistic error $O(N^{-r-1} \ln N)$, where the constants in the O-symbol depend only on r, q, M.

Let a natural number N be given. Then the functions

$$e_{\mathbf{jk}}(\mathbf{x}) = e_{j_1 k_1}(x_1) e_{j_2 k_2}(x_2),$$

$j_1, j_2 = 1, 2, \dots, N$; $k_1, k_2 = 0, 1, \dots$, form a complete orthonormal system in $L_2(\Omega)$, if one takes

$$e_{jk}(x) = \sqrt{2N} Q_k(2Nx - 2j + 1)$$

for $0 < Nx - j + 1 < 1$, and $e_{jk}(x) = 0$ otherwise, where Q_k is the normalized Legendre polynomial of degree k. Boldface denotes two-dimensional vectors, for example, $\mathbf{j} = (j_1, j_2)$, $\mathbf{k} = (k_1, k_2)$, $\mathbf{x} = (x_1, x_2)$.

Lemma. *Every function f from our class, using its values at $O(N^2)$ points, can be approximated in the norm of the space L_2 by a linear combination P of the functions $e_{\mathbf{jk}}$ corresponding to the chosen N, with accuracy up to $O(N^{-r})$, and the computation of its coefficients requires only $O(N^2)$ arithmetic operations (not counting the work of computing the values of the function f).*

This lemma can be proved with the help of the integral formula of S. L. Sobolev ([1]).

Introduce the random right-hand side

$$\Pi = P + N^{-2} \sum_{\mathbf{j}} R(\mathbf{X_j}) \sum_{\mathbf{k}} e_{\mathbf{jk}}(\mathbf{X_j}) e_{\mathbf{jk}},$$

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where $R = f - P$, $\mathbf{X_j}$ is a random point uniformly distributed in the square

$$\Omega_{\mathbf{j}} = \Omega_{j_1} \times \Omega_{j_2}, \qquad \Omega_j = [(j-1)N^{-1}, jN^{-1}],$$

and any two such points are independent; the summation is taken over those values of the indices that occur in the expression for P. Note that computation of the Fourier coefficients of the function Π with respect to the system $\{e_{\mathbf{jk}}\}$ requires $O(N^2)$ arithmetic operations.

Using the Green function of the problem under consideration and the lemma, one can prove the following theorem.

Theorem. *Let v be the solution of our boundary-value problem with right-hand side equal to Π. Then the mathematical expectation (m.e.) of the modulus of the difference between $u(\mathbf{x})$ and $v(\mathbf{x})$ is $O(N^{-r-1}\ln N)$, uniformly with respect to $\mathbf{x}$, i.e.*

$$\text{m.e.}\,|u(\mathbf{x}) - v(\mathbf{x})| \preccurlyeq N^{-r-1}\ln N. \tag{1}$$

The sign $\preccurlyeq$ means the same as O.

The equation $\Delta v = \Pi$ can be solved as follows. First, on a grid with step N^{-1}, find, with accuracy up to $O(N^{-r-2}\ln N)$, the solution $v_{\mathbf{jk}}$ of our problem with right-hand side $e_{\mathbf{jk}}$ for all values of the indices occurring in $\cdots$

expressions for Π. This can be done, for example, by means of an expansion in eigenfunctions e_p, $p_1, p_2 = 1, 2, ...$, equal to $e_p(x) = e_{p_1}(x_1)e_{p_2}(x_2)$, $e_p(x) = \sqrt{2}\sin\pi px$, and forming a complete orthonormal system in $L_2(\Omega)$. Indeed, the trigonometric Fourier coefficients $(e_{\mathbf{jk}}, e_p)$ are computed explicitly in a finite number of operations each (not depending on N). By virtue of the spectral expansion we have

$$\gamma_{\mathbf{jk}} = -\frac{1}{\pi^2}\sum \frac{e}{|\mathbf{p}|^2}(e_{\mathbf{jk}}, e_p), \tag{2}$$

where $|\mathbf{p}| = \sqrt{p_1^2 + p_2^2}$. In formula (2) retain only the terms with $|\mathbf{p}|_\infty \le N^{r+2}$, where $|\mathbf{p}|_\infty = \max(|p_1|, |p_2|)$, and denote the resulting sum by $\tilde{\gamma}_{\mathbf{jk}}$. The error from such a truncation in the uniform metric is

$$\|\gamma_{\mathbf{jk}} - \tilde{\gamma}_{\mathbf{jk}}\|_C \le \sum_{|\mathbf{p}|_\infty > N^{r+2}} |(\gamma_{\mathbf{jk}}, e_p)| \le$$

$$\le \frac{1}{\pi^2}\left(\sum_{|\mathbf{p}|_\infty > N^{r+2}} |(e_{\mathbf{jk}}, e_p)|^2 \sum_{|\mathbf{p}|_\infty > N^{r+2}} |\mathbf{p}|^{-4}\right)^{1/2} \le N^{-r-2} \tag{3}$$

uniformly in $\mathbf{j}$ and $\mathbf{k}$, since by Bessel' s inequality

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$$\sum_{|\mathbf{p}|_\infty > N^{r+2}} |(e_{\mathbf{jk}}, e_p)|^2 \leq \|e_{\mathbf{jk}}\|^2 = 1.$$

On the other hand, computing the Fourier coefficients $(e_{\mathbf{jk}}, e_p)$ with $|\mathbf{p}|_\infty \leq N^{r+2}$ requires $O(N^{2r+4})$ arithmetic operations, since there are altogether $O(N^{2r+4})$ such coefficients. Multiplying $(e_{\mathbf{jk}}, e_p)$ by $-(\pi|\mathbf{p}|)^{-2}$, we obtain the Fourier coefficients $(\gamma_{\mathbf{jk}}, e_p)$ with $|\mathbf{p}|_\infty \leq N^{r+2}$, expending another $O(N^{2r+4})$ operations. It remains to apply the fast Fourier transform [4], which makes it possible, in $O(N^{2r+4} \ln N)$ arithmetic operations, to find the values of the trigonometric polynomial

$$\tilde{\gamma}_{\mathbf{jk}} = \sum_{|\mathbf{p}|_\infty \leq N^{r+2}} (\gamma_{\mathbf{jk}}, e_p) e_p$$

on a grid with mesh $O(N^{-r-2})$. Since in the expression for Π there occur $O(N^2)$ different combinations of the indices $\mathbf{j}$ and $\mathbf{k}$, the amount of work needed to compute all functions $\gamma_{\mathbf{jk}}$ will be $O(N^{2r+6} \ln N)$ operations. We note, however, that this work can be performed once for the entire class of right-hand sides, and need not be carried out for each particular right-hand side.

If we now regard $\gamma_{\mathbf{jk}}$ as known, then v is found in $O(N^4)$ operations. Indeed, since

$$\Pi = \sum_j \sum_k (\Pi, e_{\mathbf{jk}}) e_{\mathbf{jk}}, \quad \text{then}$$

$$v = \sum_j \sum_k (\Pi, e_{\mathbf{jk}}) \gamma_{\mathbf{jk}}, \tag{4}$$

and therefore the computation of the function v at one point of a grid with mesh N^{-1} requires $O(N^2)$ operations, and on the whole grid $O(N^4)$ operations. But $\gamma_{\mathbf{jk}}$ are known to us with accuracy up to $O(N^{-r-2} \ln N)$ in the form $\tilde{\gamma}_{\mathbf{jk}}$, and therefore by formula (4) we also find v approximately, in the form $\tilde{v}$. In this case, by the triangle inequality,

$$\|v - \tilde{v}\|_C \leq \max_j \max_k \|\gamma_{\mathbf{jk}} - \tilde{\gamma}_{\mathbf{jk}}\|_C \cdot \sum_j \sum_k |(\Pi, e_{\mathbf{jk}})|. \tag{5}$$

But by the Cauchy-Bunyakovsky inequality and Parseval' s equality,

$$\sum_j \sum_k |(\Pi, e_{\mathbf{jk}})| \leq N \|\Pi\|, \tag{6}$$

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$$\|\Pi\| \leq \|P\| + \|\Pi - P\|, \qquad \|P\| \leq \|f\| + \|R\|,$$

$$\|f\| \leq \|f\|_{W_2^r} \leq \|f\|_{W_q^r} \leq M \leq 1.$$

The norm is understood in the sense of the space L_2, unless otherwise specified. It is easy to show that m.e. $\|\Pi - P\| \leqslant \|R\|$, whence m.e. $\|\Pi\| \leqslant 1 + \|R\|$. By the lemma we have $\|R\| \leqslant N^{-r} \leqslant 1$, hence m.e. $\|\Pi\| \leqslant 1$.

In view of (3), (5), and (6), m.e.

$$\|v - \bar{v}\|_C \leqslant N^{-r-1} \ln N.$$

Hence, from (1) we obtain m.e.

$$|u(\mathbf{x}) - \bar{v}(\mathbf{x})| \leqslant N^{-r-1} \ln N$$

uniformly in $\mathbf{x}$. Since $f \in C_{r-1}$, by the embedding theorem of S. L. Sobolev,

$$\|u\|_{C_{r+1}} \leqslant \|f\|_{C_{r-1}} \leqslant \|f\|_{W_q^r} \leqslant 1$$

according to the a priori estimate for our boundary-value problem (2).

Consequently, from a table with mesh size N^{-1} in each of the arguments, the solution u can be reconstructed by interpolation in a finite number of operations at any point $\mathbf{x}$ of the square Ω, with the mathematical expectation of the absolute error equal to $O(N^{-r-1} \ln N)$, uniformly in $\mathbf{x}$.

2. In the case $r = 1$ and under the additional condition

$$\sum_p |(f, e_p)|^2 p_1 p_2 \leqslant M^2$$

the amount of computational work can be reduced to $O(N^2 \ln N)$ operations. We show how to do this. We shall solve the equation $\Delta v = \Pi$ by the ordinary five-point difference scheme with mesh size N^{-1} along both axes, taking as the value of the right-hand side at a grid node the arithmetic mean of the values of the function Π over the four squares Ω_j adjacent to the given node. Note that in the case under consideration P may be regarded as a piecewise constant function; in other words, in the expression for P only e_{jk} with $k = 0$ occur. Consequently, Π will also be a piecewise constant function, more precisely $\Pi(\mathbf{x}) = f(\mathbf{X}_j)$ for $\mathbf{x} \in \Omega_j$. If the value of the solution of the resulting discrete Poisson equation at the point jN^{-1} is denoted by w_j, then, using the difference Green function, one can show that m.e.

$$|v(jN^{-1}) - w_j| \leqslant N^{-2} \ln N$$

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uniformly in j. Hence, from the theorem we obtain m.e.

$$|u(jN^{-1}) - w_j| \leqslant N^{-2} \ln N$$

uniformly in j. On the other hand, the discrete Poisson equation can be solved in $O(N^2 \ln N)$ operations by applying the fast Fourier transform (first we find the discrete Fourier coefficients of the right-hand side, then multiply them by the eigenvalues of the inverse operator of the discrete problem and, finally, perform the inverse discrete Fourier transform). Here N is assumed to be an integer power of two.

3. We now show that the method considered in item 1 differs, in the amount of information used, from the optimal method on the given class only by a factor of order $\ln N$, and that the method of item 2, in addition, differs by the same amount from the optimal one in the amount of work on the corresponding narrower class of functions. For this purpose consider the quantity

$$u(A) = \int_\Omega G(A, \mathbf{x}) f(\mathbf{x})\, d\mathbf{x}, \qquad A = (1/2, 1/2),$$

where G is the Green function of our problem. Put $g(\mathbf{x}) = G(A, \mathbf{x})$. It follows from the properties of the Green function that the function g is strictly negative and analytic in the domain $1/6 \leqslant |\mathbf{x} - A|_\infty \leqslant 1/3$; therefore, in estimating from below we may restrict ourselves to right-hand sides with support

$$|\mathbf{x} - A/2|_\infty \leqslant 1/12. \tag{7}$$

Every nondeterministic method for solving the boundary-value problem under consideration gives rise to some nondeterministic method for computing the integral

$$\int_\Omega g(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x}. \tag{8}$$

on the same class of functions f. Here g is regarded as a fixed weight. Take the collection $\{w_j\}$ of functions from § 9 of paper ([3]), reduced to the square (7), and multiply them by c/g, where c is a positive constant chosen so that

$$\max_j \|\widetilde{w}_j\|_{W_q^r} \leq M, \qquad \widetilde{w}_j = \frac{c}{g} w_j.$$

It follows from the results of paper ([3]) that if, in computing the integral (8), information about the values of the function f and its derivatives at no more than $O(N^2)$ random points is used, then the mean value of the modulus of the error for some j will be greater than $c_1 N^{-r-1}$, where c_1, like c, depends only on r, q, M. And since the integral (8) coincides with the value of the solution u of the original boundary-value problem at the center A of the square Ω, one may consider that a lower estimate has been obtained for an arbitrary nondeterministic method of solving this problem.

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The author considers it his duty to point out that he owes the formulation of the problem to N. S. Bakhvalov.

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REFERENCES

[1] S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.
[2] O. A. Ladyzhenskaya, N. N. Ural' tseva, *Linear and Quasilinear Equations of Elliptic Type*, "Nauka," 1964.
[3] N. S. Bakhvalov, in: *Numerical Methods for Solving Differential and Integral Equations and Quadrature Formulas*, "Nauka," 1964, p. 5.
[4] J. W. Cooley, J. W. Tukey, *Math. Comp.*, 19, 297 (1965).

Note: Figure translations are in progress. See original paper for figures.

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