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THEOREMS ON TRACES

1969

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Abstract

Full Text

UDC 517.5

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THEOREMS ON TRACES

AND EXTENSIONS OF FUNCTIONS FROM SURFACES

(Presented by Academician S. L. Sobolev on 28 VIII 1968)

1°. Consider functions $f(x)$, defined in n -dimensional Euclidean space R^n , with finite ν -norm

$$\|f\|_\nu = \left(\int \left| \int f(x) \nu(x, \xi) \exp(-ix\xi) dx \right|^2 d\xi \right)^{1/2}, \quad (1)$$

where $\nu(x, \xi)$ is a certain generalized function; $x = x_1, \dots, x_n$; $\xi = \xi_1, \dots, \xi_n$. The aim of the present paper is to study the values of functions $f(x)$ on $(n-1)$ -dimensional smooth surfaces Γ .

If $\nu(x, \xi) = (|\xi| + 1)^r$ and (1) defines an isotropic ν -norm coinciding with the norm W_2^r of the Sobolev space ⁽¹⁾, then the problem is easily reduced to the special case where Γ is a coordinate hyperplane, and is solved in terms of inverse embedding theorems ^(1,2). If the function ν does not depend on x , then the problem is solved when Γ is a hyperplane ⁽³⁾, and results are known for some more general surfaces Γ ⁽⁴⁾.

We specify Γ by the conditions

$$x = \kappa(y'); \quad y' = y_1, \dots, y_{n-1}; \quad -\infty < y_j < \infty, \quad j = 1, \dots, n-1. \quad (2)$$

For functions $h(y')$ defined on Γ , define a λ -norm of the form (1):

$$\|h\|_\lambda' = \left(\int \left| \int h(y') \lambda(y', \xi') \exp(-iy' \xi') dy' \right|^2 d\xi' \right)^{1/2}. \quad (1')$$

In the paper, an equality (4) is obtained which relates the functions ν and λ . The fulfillment of this equality (4) is a necessary and sufficient condition in order that every function on R^n with finite ν -norm should have on Γ a finite

λ -norm, with λ turning out to be such that every function defined on Γ with finite λ -norm can be extended from Γ to all of R^n with finite ν -norm.

In what follows we use the following notation: $Ff = \int f(x) \exp(-ix\xi) dx$ is the Fourier transform operator; $F_n[f(x)] = (F_{nf})(x', \xi_n)$ is the Fourier transform with respect to the variable x_n ; $\delta_n = \delta(x_n)$ is the δ -function in the variable x_n ;

$$A_{\nu 1}f = \int f(x)\nu(x, \xi) \exp(-ix\xi) dx; \quad A_{\nu 2}f = F^{-1}A_{\nu 1}f;$$

$C_0^\infty(R^n)$ is the set of infinitely differentiable finite functions of n variables; \mathcal{L}_2 is the space of square-summable functions;

$$(f, g) = \int f(x)\overline{g(x)} dx, \quad \|f\| = (f, f)^{1/2}.$$

Primes will denote operations on functions of $(n - 1)$ variables, for example

$$\mathcal{L}'_2 = \mathcal{L}_2(R^{n-1}), \quad A'_{\lambda 1}h = \int h(x')\lambda(x', \xi') \exp(-ix'\xi') dx';$$

$$(f, g)' = \int f(x')\overline{g(x')} dx'.$$

I' denotes proposition I in which n is replaced by $(n - 1)$.

The space H_ν is the set of functions from \mathcal{L}_2 with finite ν -norm (1). We impose on the function $\nu(x, \xi)$ two basic requirements:

I. For some natural number r and any function from the Sobolev space W_2^r , the norm (1) is finite.

II. The equation $A_{\nu 1}f = g$ is uniquely solvable in \mathcal{L}_2 for any function $g \in \mathcal{L}_2$.

When conditions I, II are fulfilled, we shall call the function ν admissible (respectively λ -admissible, if I' , II' are fulfilled). From conditions I, II there follows the embedding $\mathcal{L}_2 \supset H_\nu \supset W_2^r$ together with the topology and the density of W_2^r and C_0^∞ in H_ν .

Let $x = \varphi(y)$ be a smooth change of variables taking Γ into the hyperplane $y_n = 0$; more precisely, it is assumed that $\varphi(y', 0) = \chi(y')$, and if $D\varphi = |(\partial\varphi_k/\partial y_j)|$ is the Jacobian of the change, then $0 < c_1 \leq D\varphi \leq c_2 < \infty$. As a result of this change the space H_ν passes into H_ω , where

$$\omega(y, \xi) = \nu(\varphi(y), \xi) \exp[i(y - \varphi(y))\xi] D\varphi(y).$$

Clearly, the function $\omega(y, \xi)$ is again admissible. Therefore in H_ν -spaces it suffices for us to consider the special case when Γ is the hyperplane $x_n = 0$.

Let $\Pi[f(x)] = f(x', 0)$. Then our problem can be formulated as follows: indicate conditions under which a pair of admissible functions $\nu(x, \xi)$ and $\lambda(x', \xi')$ defines spaces H_ν and H'_λ ; the operator $\Pi : H_\nu \rightarrow H'_\lambda$ is bounded and acts onto all of H'_λ . A necessary condition for the solvability of this problem is

III. $H_\nu^* \supset a(x')\delta(x_n)$ for all $a(x') \in \mathcal{L}'_2$, where H_ν^* is the space adjoint to H_ν .

We shall say that an admissible function $\mu(x', \xi')$ defines an equivalent norming of the space H'_λ , if $H'_\lambda = H'_\mu$. On the set $C_0^\infty(R^{n-1})$ define the operator

$$B \equiv B_{\nu\mu} = A_{\nu 2}^{-1} A_{\nu 2}^{-1*} \delta_n A_{\mu 2}^* : C_0^\infty(R^{n-1}) \rightarrow H_\nu. \quad (3)$$

The main result of the work is now formulated in the following two assertions.

Theorem 1. The operator Π maps all of H_ν onto all of H'_λ if and only if in the space H'_λ one can prescribe an equivalent norming by a function $\mu(x', \xi')$ such that, for the pair of functions (ν, μ) , on the set $C_0^\infty(R^{n-1})$ the relation

$$\Pi B_{\nu\mu} = I' \quad \text{—the identity operator} \quad (4)$$

holds.

Theorem 2. a) If μ is given with properties I', I'', then equation (4) with respect to ν has a solution with properties I, II, III.

b) If ν is given with properties I, II, III, then equation (4) with respect to μ has a solution with properties I', II'.

2°. We shall prove the sufficiency in Theorem 1, then Theorem 2, and then the necessity in Theorem 1.

Sufficiency of condition (4) in Theorem 1. Let $h(x') \in C_0^\infty(R^{n-1})$ and $f(x) = Bh$. Then

$$\begin{aligned} \|f\|_\nu^2 &= \|A_{\nu 2} Bh\|^2 = \|A_{\nu 2}^{-1*} \delta_n A_{\mu 2}^* A'_{\mu 2} h\|^2 \\ &= (\delta_n A_{\mu 2}^* A'_{\mu 2} h, Bh) = (A_{\mu 2}^* A'_{\mu 2} h, \Pi Bh)' = \|h\|_\mu'^2. \end{aligned}$$

Thus, the operator B , acting from H'_μ into H_ν , is bounded and can be extended by continuity to all of H'_μ . From condition III it follows that on all of H_ν the operator Π is defined with range in \mathcal{L}'_2 .

Now take an arbitrary element $h \in H'_\mu$ and consider $f = Bh$ and $\Pi f = \Pi Bh$. The operator $\Pi B : H'_\mu \rightarrow \mathcal{L}'_2$ is bounded and on the set $C_0^\infty(R^{n-1})$, dense in H'_μ , coincides with the identity I' . Therefore $\Pi B = I'$ on all H'_μ , i.e. for every element $h \in H'_\mu$ there exists $f = Bh \in H_\nu$ for which $\Pi f = h$.

Conversely, let $f(x) \in H_\nu$. It is necessary to show that $\Pi f = h \in H'_\mu$. Note that $\Pi(f - Bh) = 0$ and, as we have shown, $\|h\|'_\mu = \|Bh\|_\nu$. We have

$$\|f\|_\nu^2 = \|A_{\nu 2} [(f - Bh) + Bh]\|^2 =$$

$$\begin{aligned} &= \|A_{\nu_2}(f - Bh)\|^2 + \|A_{\nu_2}Bh\|^2 + 2\operatorname{Re}(A_{\nu_2}(f - Bh), A_{\nu_2}Bh) \geq \\ &\geq \|A_{\nu_2}Bh\|^2 + 2\operatorname{Re}((f - Bh), \delta_n A_{\mu_2}^* A'_{\mu_2} h) = \|h\|_{\mu}^2, \end{aligned}$$

which was required to be proved.

Proof of Theorem 2. a). Let $\pi(\xi_n)$ be a smooth function satisfying $1 \leq \pi(\xi_n) \leq (|\xi_n| + 1)^r$ for some $r > 0$,

$$\int \pi^{-2}(\xi_n) d\xi_n = 1.$$

For a given admissible function $\mu(x', \xi')$, construct $\nu(x, \xi) = \mu(x', \xi')\pi(\xi_n)$, which is again admissible. If $f \in H_{\nu}$, then

$$\begin{aligned} A_{\nu_1} f &= \int (F_{nf})(x', \xi_n) \mu(x', \xi') \pi(\xi_n) \exp(-ix' \xi') dx' = g(\xi) \in \mathcal{L}_2, \\ A'_{\mu_1} \Pi f &= \int d\xi_n \int (F_{nf})(x', \xi_n) \mu(x', \xi') \exp(-ix' \xi') dx' \\ &= \int g(\xi) \pi^{-1}(\xi_n) d\xi_n \in \mathcal{L}'_2. \end{aligned}$$

Thus, $\Pi f \in H'_{\mu}$. Therefore for all $a(x') \in \mathcal{L}'_2 \subset H'_{\mu}$, $(\Pi f, a)' = (f, \delta_n a)$, i.e. $a(x')\delta(x_n) \in H_{\nu}^*$, and III holds. Let us verify that relation (4) is satisfied:

$$\begin{aligned} A_{\nu_2}^{-1} &= F_n^{-1} \pi^{-1}(\xi_n) F_n A'_{\mu_2}^{-1}, \quad B = F_n^{-1} A_{\mu_2}^{-1} A'^{-1}_{\mu_2} \pi^{-2}(\xi_n) F_n \delta_n A_{\mu_2}^* A'_{\mu_2} = \\ &= F_n^{-1} \pi^{-2}(\xi_n), \\ \Pi B h &= \int d\xi_n \pi^{-2}(\xi_n) h = h. \end{aligned}$$

b) Transform relation (4). For all $f(x) \in C_0^{\infty}(R^n)$ we have

$$\Pi A_{\nu_2}^{-1} A_{\nu_2}^{-1} \delta_n A_{\mu_2}^* A'_{\mu_2} \Pi f = \Pi f,$$

or

$$h(x') = A_{\mu_2}^* A'_{\mu_2} \Pi f, \quad g(x) = A_{\nu_2}^{-1} A_{\nu_2}^{-1} \delta_n h, \quad \Pi g = \Pi f. \quad (5)$$

We must find an admissible μ such that, for all $f \in C_0^{\infty}(R^n)$, the system of equations (5) is compatible with respect to $h \in H_{\mu}^*$ and $g \in H_{\nu}$. If μ is found, then by conditions I', II' it is necessary that $h \in \mathcal{L}'_2$. Therefore, considering (5) as a system with respect to f and g from H_{ν} for a fixed function $h \in \mathcal{L}'_2$, we impose on μ more stringent restrictions than (4). We shall seek μ under these new conditions, assuming $h \in \mathcal{L}'_2$. Then g lies in the range R of the operator $A_{\nu_2}^{-1} A_{\nu_2}^{-1} \delta_n : \mathcal{L}'_2 \rightarrow H_{\nu}$, restricted by virtue of III. The operator Π , acting from $R \subset H_{\nu}$ to \mathcal{L}'_2 , is bounded:

$$\|\Pi\| = \sup_g (\|\Pi g\|' / \|g\|_{\nu}) = \sup_{g, h} (|(g, \delta_n h)| / \|g\|_{\nu} \|h\|') \leq \sup_h (\|\delta_n h\|_{\nu}^* / \|h\|') < \infty$$

(by condition III).

The range D of the operator $\Pi : R \rightarrow \mathcal{L}'_2$ is dense in \mathcal{L}'_2 : if $0 = (\psi, \Pi g)'$ for all $g \in R$, then $0 = [\psi, \Pi A_{\nu_2}^{-1} A_{\nu_2}^{-1*} \delta_{nh}]'$ for all $h \in \mathcal{L}'_2$, and, taking $h = \psi$, we obtain $0 = A_2^{-1*} \delta_n \psi$, i.e. $\psi = 0$. On R , Π has an inverse $\Pi^{-1} : \mathcal{L}'_2 \supset D \rightarrow R$.

Indeed, if $g \in R$ and $\Pi g = 0$, then for all $\varphi \in \mathcal{L}'_2$

$$0 = (\varphi, \Pi g)' = (\varphi \delta_n, g) = (\varphi \delta_n, A_{\nu_2}^{-1} A_{\nu_2}^{-1*} h \delta_n) = (A_{\nu_2}^{-1*} \varphi \delta_n, A_{\nu_2}^{-1*} h \delta_n).$$

Taking $\varphi = h$, we obtain $A_{\nu_2}^{-1*} h \delta_n = 0$, i.e. $h = 0$ and $g = 0$. Thus, if $g \in R$, then $\Pi g = u \in D$ and $g = \Pi^{-1} u$. The second equation of system (5) is written in the form

$$h = F_n A_{\nu_2}^* A_{\nu_2} g = F_n A_{\nu_2}^* A_{\nu_2} \Pi^{-1} u.$$

Thus, we seek the function μ from the compatibility conditions, for all $h \in \mathcal{L}'_2$, of the system of equations with respect to $u \in D$:

$$h = A_{\mu_2}^* A_{\mu_2}' u, \quad h = F_n A_{\nu_2}^* A_{\nu_2} \Pi^{-1} u. \quad (6)$$

Consider

$$P = F_n A_{\nu_2}^* A_{\nu_2} \Pi^{-1} : \mathcal{L}'_2 \supset D \rightarrow \mathcal{L}'_2.$$

There exists

$$P^{-1} = \Pi A_{\nu_2}^{-1} A_{\nu_2}^{-1*} \delta_n : \mathcal{L}'_2 \rightarrow D$$

and

$$(P^{-1} h, h) = \|A_{\nu_2}^{-1*} \delta_n h\|^2 \geq 0.$$

Thus, P —

positive operator, and from it one can extract the square root

$$\sqrt{P} : D \rightarrow \mathcal{L}'_2. \quad (5)$$

The domain of definition of \sqrt{P} can be extended to the set $D_{\sqrt{P}}$ by adding all those elements of \mathcal{L}'_2 which \sqrt{P} maps into \mathcal{L}'_2 . $D_{\sqrt{P}}$, with norm $\|h\|'_{\sqrt{P}} = \|\sqrt{P} h\|'$, becomes a Hilbert space $H'_{\sqrt{P}}$, and for all $\Pi f \in C_0^\infty(R^{n-1})$ condition (4) is fulfilled:

$$\Pi A_{\nu_2}^{-1} A_{\nu_2}^{-1*} \delta_n \sqrt{P} \sqrt{P} \Pi f = \Pi f.$$

Therefore the operator Π maps all of H_ν onto all of $H'_{\sqrt{P}}$, and, according to condition I and the embedding theorems for $W_2^r(R^n)$, we have the embedding

$$H'_{\sqrt{P}} \supset W_2^{r-1/2}(R^{n-1})$$

together with the topology. This means that the operator

$$K = \sqrt{\overline{P}} F'^{-1} (|\xi'| + 1)^{-r+1/2} F' : \mathcal{L}'_2 \rightarrow \mathcal{L}'_2 \quad (5)$$

is bounded and is a Hilbert-Schmidt operator.

Therefore there is a function $k(x', y') \in \mathcal{L}_2(R^{2n-2})$ such that

$$(Kh)(x') = \int k(x', y') h(y') dy'.$$

Thus, also for the operator $\sqrt{\overline{P}}$ we can write integral representations

$$\begin{aligned} (\sqrt{\overline{P}}h)(x') &= KF'^{-1} (|\xi'| + 1)^{r-1/2} F'h \\ &= \int (F_{y'}k)(x', \xi') (|\xi'| + 1)^{r-1/2} (F'h)(\xi') d\xi', \end{aligned}$$

first defined on $W_2^{r-1/2}(R^{n-1})$, and then extended by continuity to all of $H'_{\sqrt{\overline{P}}}$.

Let

$$\mu(z', \eta') = (2\pi)^{-n} \int dx' \int d\xi' \int dy' k(x', y') (|\xi'| + 1)^{r-1/2} \exp\{i\xi'(z' - y') + i\eta'(z' - x')\}.$$

Then the operator $F'^{-1}\sqrt{\overline{P}}h$ can be written as

$$(F'^{-1}\sqrt{\overline{P}}h)(\xi') = \int h(x') \mu(x', \xi') \exp(-ix'\xi') dx'.$$

Let us now note that $H'_{\sqrt{\overline{P}}} = H'_\mu$, and that the function μ satisfies conditions I', II'. This proves Theorem 2.

Necessity of condition (4) in Theorem 1. It is known that Π maps all of H'_ν onto all of H'_λ . For the given ν , according to Theorem 2, we find some μ . Then for every $h \in H'_\lambda$ there exists an extension f in H'_ν , and the trace $\Pi f = h$ lies in H'_μ . Hence $H'_\mu \subset H'_\lambda$. But in exactly the same way every h from H'_μ lies in H'_λ . Therefore $H'_\mu = H'_\lambda$.

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Received
12 VI 1968

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Note: Figure translations are in progress. See original paper for figures.

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