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Abstract

Full Text

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ON SATURATED AND COMPLETELY SATURATED SUBSPACES OF CERTAIN COORDINATE SPACES

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In connection with certain problems of analysis and, first of all, approximation theory, interest arose in describing subspaces E_0 of a Banach space E having the property that from the condition

$$Ax \equiv x \quad (x \in E_0), \tag{1}$$

where A is a linear operator leaving invariant some convex set V , there follows the identity

$$Ax \equiv x \quad (x \in E). \tag{2}$$

Descriptions of certain classes of such subspaces in the case when V is a cone in the space E are contained in works ⁽¹⁻⁵⁾, etc. In the case when V is the unit ball of the space E , such subspaces (we called them subspaces possessing the e -property) were studied in works ⁽⁴⁻⁶⁾. Thus, in ⁽⁶⁾ a description was given of a certain class of finite-dimensional subspaces possessing the e -property in the spaces l_p^n , l_p , and L_p .

Alongside this, it is of interest to single out the following class of subspaces possessing stronger properties. Let E_0 be a subspace of a Banach space E , and suppose that from the conditions

$$\|A_n x\| \leq \|x\| \quad (x \in E), \quad \lim_{n \rightarrow \infty} \|A_n x - x\| = 0 \quad (x \in E_0), \tag{3}$$

where A_n is a sequence of linear operators, there follows the identity

$$\lim_{n \rightarrow \infty} \|A_n x - x\| = 0 \quad (x \in E). \tag{4}$$

Then we shall say that the subspace E_0 possesses the **complete e -property**. A number of theorems on subspaces possessing the complete e -property were established in works ^(4,5).

In the present article an attempt is made to describe finite-dimensional subspaces possessing the e -property and the complete e -property in the coordinate spaces c, m^n, λ_α^n , where λ_α^n is the n -dimensional space with the Lorentz norm ⁽⁷⁾.

1. Important classes of subspaces possessing the e -property and the complete e -property are, respectively, the classes of saturated and completely saturated subspaces (see ⁽⁴⁻⁶⁾). Let us recall their definition.

Let S and S^* be the unit spheres, respectively, in the Banach space E and its conjugate E^* . It is said that a functional $f \in S^*$ passes through a point $x \in S$ if $f(x) = 1$. A point $x_0 \in S$ is called a smooth point of S if through it there passes a unique functional $f_0 \in S^*$. It is said that a subspace E_0 of the space E is saturated by smooth points of S if the set $F(E_0)$ of functionals passing through smooth points of S lying in E_0 is total. Such subspaces are called saturated.

It is said that E_0 is completely saturated if the following conditions are fulfilled: a) E_0 is a saturated subspace; b) the set $F(E_0)$ contains such a total subset $F_0(E_0)$ that the norm defined by the equality

$$\|x\|^* = \sup_{f \in F_0(E_0)} |f(x)|, \quad (5)$$

is equivalent to the original norm of the space E ; c) every sequence of functionals $f_n \in F_0(E_0)$ has a limit point $f_0 \in F(E_0)$ in the weak topology.

2. Let us first consider the n -dimensional space m^n with norm $\|x\| = \max_{1 \leq i \leq n} |\xi_i|$.

Theorem 1. For linearly independent vectors

$$e_1 = \{\xi_{11}, \dots, \xi_{1n}\}, \dots, e_k = \{\xi_{k1}, \dots, \xi_{kn}\} \quad (6)$$

to form a basis of a saturated subspace E_k in m^n , it is necessary and sufficient that each point of the collection $M_k = \{\pm g_i\}_1^n$, where

$$g_i = \{\xi_{1i}, \xi_{2i}, \dots, \xi_{ki}\} \quad (i = 1, \dots, n), \quad (7)$$

not belong to the convex hull of the others.

It follows from Theorem 1 that two-dimensional saturated subspaces exist in the space m^n . Indeed, if M_2 is a plane centrally symmetric convex $2n$ -gon, then to every collection of n distinct vertices $\{x_i, y_i\}$ ($i = 1, \dots, n$) of it, containing no opposite vertices, there correspond vectors

$$\{x_1, x_2, \dots, x^n\} \in m^n, \quad \{y_1, y_2, \dots, y_n\} \in m^n,$$

forming a basis of a two-dimensional saturated subspace of the space m^n .

Let us note that every k -dimensional subspace containing a saturated subspace of smaller dimension is saturated. If, however, a subspace does not contain saturated subspaces of minimal dimension, then the question of whether it can be saturated is resolved differently for various concrete spaces. Thus, in the space l_p^n (see (6)) there are no saturated subspaces that do not contain saturated subspaces of minimal dimension, while in the space m^n such subspaces may exist. For example, the subspace E_3 spanned by the vectors

$$e_1 = \{1, 1, 1, 1\}, \quad e_2 = \{1, 1, -1, -1\}, \quad e_3 = \{1, -1, 1, -1\},$$

is a saturated subspace of the space m^4 , since the convex hull M_3 is a cube in three-dimensional space and the conditions of Theorem 1 are fulfilled. However, none of the projections of the cube can be an octagon, and therefore E_3 , as follows from the following Theorem 2, does not contain two-dimensional saturated subspaces of the space m^4 .

Theorem 2. For a saturated subspace E_k of the space m^n , spanned by the vectors (6), to contain a saturated subspace E_l of smaller dimension ($1 < l < k$), it is necessary and sufficient that there exist a projection of the convex hull M_k onto some l -dimensional subspace having the same number of vertices.

3. Let us turn to the consideration of the space c of convergent sequences, in which the norm is defined by the equality $\|x\| = \sup_i |\xi_i|$.

Introduce the notation

$$\xi_0(x) = \xi_0 = \lim_{i \rightarrow \infty} \xi_i \quad (x = \{\xi_1, \xi_2, \dots\} \in c). \quad (8)$$

Theorem 3. Let E_k be a k -dimensional subspace of the space c , and let the vectors

$$e_1 = \{\xi_{11}, \xi_{12}, \dots\}, \dots, e_k = \{\xi_{k1}, \xi_{k2}, \dots\} \quad (9)$$

form a basis in E_k .

E_k is a saturated subspace if and only if all vectors of the set $M = \{\pm g_i\}_0^\infty$, where

$$g_i = \{\xi_{1i}, \xi_{2i}, \dots, \xi_{ki}\} \quad (i = 0, 1, 2, \dots), \quad (10)$$

are distinct extreme points of the convex hull of M .

Thus, for example, the subspace spanned by the vectors

$$e_1 = \left\{ -1, 0, \frac{1}{2}, \dots, \frac{n-2}{n-1}, \dots \right\},$$

$$e_2 = \left\{ 1, 2, 1 + \frac{\sqrt{3}}{2}, \dots, 1 + \frac{\sqrt{2n-3}}{n-1}, \dots \right\}, \quad (11)$$

is a saturated subspace in c .

From the results obtained in (8), it follows that in the space c_0 of sequences converging to zero there are no finite-dimensional saturated subspaces. We note that c_0 likewise contains no finite-dimensional saturated subspaces of the space c , although it is itself a saturated subspace in c .

A description of completely saturated finite-dimensional subspaces of the space c is given by Theorem 4. By an exposed point g of a convex set G one means an extreme point of G through which there passes a hyperplane supporting G and having with G only g in common (see (9)).

Theorem 4. Let E_k be a k -dimensional subspace of the space u , and let the vectors (9) form a basis in E_k .

E_k is a completely saturated subspace if and only if all vectors of the set $M = \{\pm g_i\}_0^\infty$, where g_i is defined by formula (10), are distinct exposed points of the convex hull of M .

Thus, for example, the subspace spanned by the vectors

$$e_1 = \left\{ 0, \frac{1}{2}, \dots, \frac{n-1}{n}, \dots \right\}, \quad e_2 = \left\{ 2, 1 + \frac{\sqrt{3}}{2}, \dots, 1 + \frac{\sqrt{2n-1}}{n}, \dots \right\}$$

obtained from the vectors (11) by a shift by one coordinate, is completely saturated, whereas the subspace spanned by the vectors (11) is not a completely saturated subspace of the space c .

4. Consider the n -dimensional spaces with the Lorentz metric λ_α^n ($0 < \alpha < 1$). Let $x = \{\xi_1, \xi_2, \dots, \xi_n\} \in \lambda_\alpha^n$. The norm in the space λ_α^n is given by the formula

$$\|x\|_{\lambda_\alpha^n} = \int_0^\infty \varphi^\alpha(x, h) dh, \quad (12)$$

where $\varphi(x, h) = \text{mes}\{s : |\xi_s| \geq h\}$.

The following Theorem 5 shows that saturated subspaces form, in a certain sense, a dense set in the set of all subspaces of the space λ_α^n .

Theorem 5. Let E_0 be a subspace of λ_α^n of dimension greater than 1, and let ε be an arbitrary positive number.

There exists a saturated subspace E'_0 in λ_α^n such that the gap (see (10)) between the subspaces E_0 and E'_0 satisfies the inequality $\theta(E_0, E'_0) < \varepsilon$.

In the cases $k = 3$ and $n = 4$ one can give a complete description of saturated subspaces of the space λ_α^n . Thus, in λ_α^4 every subspace of dimension greater than 1 that contains at least one point of smoothness of the unit

of the sphere S , is saturated. For $n = 3$ the exceptional case is that in which $\alpha = \alpha_0 = \log_3 2$. There, if $\alpha \neq \alpha_0$, then again every two-dimensional subspace in λ_α^3 containing at least one smooth point of the unit sphere is saturated. If, however, $\alpha = \alpha_0$, then in order that the two-dimensional subspace spanned by the vectors

$$e_1 = \{\xi_{11}, \xi_{12}, \xi_{13}\}, \quad e_2 = \{\xi_{21}, \xi_{22}, \xi_{23}\}$$

be saturated, it is necessary and sufficient that for the vectors $g_i = \{\xi_{1i}, \xi_{2i}\}$ ($i = 1, 2, 3$) and arbitrary δ_i , taking one of the values 0, 1, -1, from the condition $\sum_{i=1}^3 |\delta_i| > 0$ it follow that $\sum_{i=1}^3 \delta_i g_i \neq 0$.

Thus, for example, the subspace spanned by the vectors

$$e_1 = \{1, 0, -1\}, \quad e_2 = \{0, 1, 0\}$$

is not saturated in λ_α^3 for any value $\alpha \in (0, 1)$; the subspace spanned by the vectors

$$e'_1 = \{1, 0, -1\}, \quad e'_2 = \{1, 1, 0\},$$

is saturated in λ_α^3 for all values $\alpha \in (0, 1)$, except $\alpha = \alpha_0$, and the subspace spanned by the vectors

$$e''_1 = \{1, 0, -1\}, \quad e''_2 = \{1, 1, 1\},$$

is saturated in λ_α^3 for all values $\alpha \in (0, 1)$.

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