ON THE DISTRIBUTION OF VALUES OF MULTIPLICATIVE ARITHMETIC FUNCTIONS

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Abstract

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ON THE DISTRIBUTION OF VALUES OF MULTIPLICATIVE ARITHMETIC FUNCTIONS

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1. One of the problems of the probabilistic theory of numbers is the study of conditions for the convergence of distribution functions (d.f.)

$$\nu_n\{g(m) < x\} = \frac{1}{n} \sum_{\substack{g(m)<x \\ m \le n}} 1, \tag{1}$$

where $g(m)$ is a real multiplicative arithmetic function, i.e. $g(mn) = g(m)g(n)$ for all relatively prime m, n. For this purpose a very suitable tool is the characteristic transform proposed by V. M. Zolotarev ([1]). The characteristic transform (c.t.) of a certain d.f. $F(x)$ is the diagonal matrix of the second order $W(t)$ with elements $\{w_0(t), w_1(t)\}$ on the main diagonal, where

$$w_j(t) = \int_{-\infty}^{+\infty} |x|^{it} \operatorname{sgn}^j x \, dF(x), \qquad j = 0, 1,$$

with $0^{it} = 0$ for any real t. Suppose that, as $n \to \infty$, the d.f.' s $F_n(x)$ converge weakly to some d.f. $F(x)$. Since the kernel of the c.t. has a discontinuity at the point $x = 0$, for a continuous correspondence between the d.f.' s $F_n(x)$ and the corresponding c.t.' s $W_n(t)$, to the usual conditions of Lévy' s theorem it is necessary to add the following: $F_n(0) \to F(0)$, $F_n(+0) \to F(+0)$ as $n \to \infty$.

Since for the d.f. (1) we have

$$w_j(t) = \frac{1}{n} \sum_{m \le n} |g(m)|^{it} \operatorname{sgn}^j g(m), \qquad j = 0, 1,$$

the question of the existence of a limiting distribution in the sense indicated above can be solved with the aid of theorems on the summation of multiplicative arithmetic functions ([2]). In this way the following results are obtained.

Theorem 1. The d.f.' s (1), as $n \to \infty$, converge to a certain asymmetric limiting d.f. if and only if the series

$$\sum_p \frac{(|g(p)|-1)^*}{p}, \tag{2}$$

$$\sum_p \frac{((|g(p)|-1)^*)^2}{p}, \tag{3}$$

$$\sum_{g(p)<0} \frac{1}{p} \tag{4}$$

converge, and there exists a natural number α such that $g(2^\alpha) \neq -1$, where p is a prime number, and the asterisk denotes truncation at level 1, i.e.

$$x^* = \begin{cases} x, & \text{if } |x| \le 1, \\ 1, & \text{if } |x| > 1. \end{cases}$$

The elements of the characteristic function of the limiting distribution law are equal to

$$w_j(t) = \prod_p \left(1 - \frac{1}{p}\right) \sum_{\alpha=0}^{\infty} \frac{|g(p^\alpha)|^{it} \operatorname{sgn}^j g(p^\alpha)}{p^\alpha}, \qquad j = 0, 1.$$

Erdős' theorem ([3,4]) on the distribution of values of nonnegative multiplicative arithmetic functions is a special case of Theorem 1. Moreover, the convergence conditions for the series (2), (3), (4) are equivalent to the following: for some $c > 0$ the series

$$\sum_{|\ln|g(p)||>c} \frac{1}{p}, \qquad \sum_{|\ln|g(p)||\le c} \frac{\ln|g(p)|}{p}, \qquad \sum_{|\ln|g(p)||\le c} \frac{\ln^2|g(p)|}{p}$$

converge. Therefore, putting in Theorem 1 $g(m) = \exp\{f(m)\}$, where $f(m)$ is a real additive arithmetic function, i.e. $f(mn) = f(m) + f(n)$ for all coprime m, n, we obtain as a special case the well-known Erdős-Wintner theorem ([5,6]) on the distribution of values of additive functions.

2. In the case when the series (2) diverges while the series (3) converges, the multiplicative function $g(m)$ must be "centered" by multiplying it by a certain constant a_n, $n = 1, 2, \ldots$. By applying some results of G. Delange ([7]), the following has been obtained.

Theorem 2. Let

$$a_n = \exp\left\{-\sum_{p\le n}\frac{(|g(p)|-1)^*}{p}\right\}$$

and, as $x\to\infty$, for every $\lambda>1$,

$$\sum_{x<p\le x^\lambda}\frac{(|g(p)|-1)^*}{p}\to 0. \qquad (5)$$

The distribution functions d.f. $\nu_n\{a_n g(m)<x\}$ as $n\to\infty$ converge to a certain limiting nonsymmetric distribution function if and only if the series (3), (4) converge and there exists a natural number α such that $g(2^\alpha)\ne -1$.

The limiting distribution law is continuous if and only if $g(m)\ne 0$ and the series

$$\sum_{|g(p)|\ne 1}\frac{1}{p}$$

diverges.

The elements of the characteristic function corresponding to the limiting distribution law are equal to

$$w_j(t)=\lim_{n\to\infty}a_n^{it}\prod_{p\le n}\left(1-\frac{1}{p}\right)\sum_{\alpha=0}^{\infty}\frac{|g(p^\alpha)|^{it}\operatorname{sgn}^j g(p^\alpha)}{p^\alpha}, \qquad j=0,1.$$

We note that, since by Cauchy' s inequality we have

$$\left|\sum_{x<p\le x^\lambda}\frac{(|g(p)|-1)^*}{p}\right|\le\left(\sum_{x<p\le x^\lambda}\frac{((|g(p)|-1)^*)^2}{p}\right)^{1/2}\left(\sum_{x<p\le x^\lambda}\frac{1}{p}\right)^{1/2},$$

condition (5) is not required for the proof of the sufficiency in Theorem 2.

3. If the series (3) diverges, then the probabilistic interpretation of arithmetic functions developed by I. P. Kubilius ([8]) may be applied; it makes it possible, instead of distributions of values of multiplicative arithmetic functions, to consider distributions of products of certain random variables. We applied this method to strongly multiplicative arithmetic functions, i.e. to such multiplicative functions $g(m)$ for which, for all natural α and primes p, we have $g(p^\alpha)=g(p)$. Moreover, we restrict ourselves to the class of strongly multiplicative arithmetic functions satisfying the conditions (class $\mathfrak{M}_H$): as $n\to\infty$

1)

$$B(n)=\left(\sum_{\substack{p\le n\\ g(p)\ne 0}}\frac{\ln^2|g(p)|}{p}\right)^{1/2}\to\infty;$$

2) there exists an unbounded increasing function $r=r(n)$ such that

$$\ln r/\ln n\to 0,\qquad B(r)/B(n)\to 1;$$

3)

$$\nu_n\left\{\prod_{\substack{p|m\\ p>r}}g(p)<0\right\}\to 0.$$

For functions of this class we consider the d.f.

$$\nu_n\{a_n|g(m)|^{b_n}\operatorname{sgn}g(m)<x\},\tag{6}$$

where

$$a_n=\exp\left\{-\frac{1}{B(n)}\sum_{\substack{p\le n\\ g(p)\ne 0}}\frac{\ln|g(p)|}{p}\right\},\qquad b_n=\frac{1}{B(n)}.$$

If $w_j(t)$, $j=0,1$, are elements of the ch.f. of a certain distribution law, then

$$\lambda_j(m)=(-i)^m\left[\frac{d^m}{dt^m}\ln w_j(t)\right]_{t=0},\qquad j=0,1,$$

are called the M-semi-invariants of this law.

Theorem 3. For convergence of the d.f. (6), where $g(m)\in\mathfrak{M}_H$, to a proper limiting d.f. with M-semi-invariants $\lambda_0(2)=1$, $\lambda_1(2)=\lambda_1$ ($|\lambda_1|\le 1$), when it is asymmetric, and $\lambda_0(2)=1$, when it is symmetric, it is necessary and sufficient that the following conditions be fulfilled.

In the case

$$\beta=\prod_{g(p)<0}\left(1-\frac{2}{p}\right)>0:$$

a)

$$\sum_{g(p)=0}\frac{1}{p}<+\infty;$$

b) there exist nondecreasing functions $K^{(\nu)}(x)$, $K^{(\nu)}(-\infty) = 0$, $\nu = 1, 2$, with variations

$$\operatorname{Var} K^{(\nu)}(x) = \frac{1}{2}[1 - (-1)^\nu \lambda_1], \qquad \nu = 1, 2,$$

where $K^{(2)}(x)$ satisfies the condition

$$\exp\left\{-2\int_{-\infty}^{+\infty} \frac{1}{x^2}\, dK^{(2)}(x)\right\} \geq \beta,$$

such that the integral

$$\int_{-\infty}^{+\infty} \frac{1}{x}\, dK^{(2)}(x) = c$$

converges and, as $n \to \infty$,

1)
$$\frac{1}{B(n)} \sum_{\substack{p \leq n \\ g(p) < 0}} \frac{\ln |g(p)|}{p} \to c;$$

2)
$$\frac{1}{B^2(n)} \sum_{\substack{p \leq n \\ g(p) > 0 \\ \ln g(p) < xB(n)}} \frac{\ln^2 g(p)}{p} \to K^{(1)}(x),$$

3)
$$\frac{1}{B^2(n)} \sum_{\substack{p \leq n \\ g(p) < 0 \\ \ln |g(p)| < xB(n)}} \frac{\ln^2 |g(p)|}{p} \to K^{(2)}(x),$$

where convergence takes place not only at every point of continuity of the functions $K^{(\nu)}(x)$, $\nu = 1, 2$, respectively, but also at the point $x = +\infty$.

In the case $\beta = 0$, condition a) is retained, while b) is replaced by the following: there exists a nondecreasing function $K(x)$, $K(-\infty) = 0$, with variation 1, such that as $n \to \infty$

$$\frac{1}{B^2(n)} \sum_{\substack{p \leq n \\ g(p) \neq 0 \\ \ln |g(p)| < xB(n)}} \frac{\ln^2 |g(p)|}{p} \to K(x)$$

not only at every point of continuity of the function $K(x)$, but also at the point $x = +\infty$.

If condition a) is not fulfilled, then for any constants $a_n > 0$, $b_n > 0$, the d.f. (6) converge to an improper d.f.

To the limiting d.f. there corresponds a characteristic function with components

$$w_0(t) = \prod_{g(p)=0} \left(1 - \frac{1}{p}\right) f^{(1)}(t) f^{(2)}(t), \qquad w_1(t) = \beta \prod_{g(p)=0} \left(1 - \frac{1}{p}\right) \frac{f^{(1)}(t)}{f^{(2)}(t)},$$

$$f^{(1)}(t) = \exp\left\{-ict + \int_{-\infty}^{+\infty} (e^{itx} - 1 - itx) \frac{1}{x^2}\, dK^{(1)}(x)\right\},$$

$$f^{(2)}(t) = \exp\left\{\int_{-\infty}^{+\infty} (e^{itx} - 1) \frac{1}{x^2}\, dK^{(2)}(x)\right\},$$

or

$$w_0(t) = \prod_{g(p)=0} \left(1 - \frac{1}{p}\right) \exp\left\{\int_{-\infty}^{+\infty} (e^{itx} - 1 - itx) \frac{1}{x^2}\, dK(x)\right\}, \qquad w_1(t) \equiv 0,$$

depending on whether $\beta > 0$ or $\beta = 0$.

By choosing the corresponding functions $K^{(\nu)}(x)$, $\nu = 1, 2$, from this theorem we obtain conditions of convergence to the given distribution laws. In particular, if

$$K^{(1)}(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } x > 0, \end{cases} \qquad K^{(2)}(x) \equiv 0,$$

then we find conditions for convergence of the d.f. (6) to the following logarithmically normal law:

$$F(\alpha_0, \alpha_1, x) = \begin{cases} \dfrac{\alpha_0 - \alpha_1}{2} \left[1 - G(\ln|x|)\right] & \text{for } x < 0, \\ 1 - \dfrac{\alpha_0 + \alpha_1}{2} \left[1 - G(\ln x)\right] & \text{for } x > 0, \end{cases}$$

where $0 < \alpha_0 \leq 1$, $|\alpha_1| \leq \alpha_0$,

$$G(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2}\, dt.$$

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Note: Figure translations are in progress. See original paper for figures.