A CRITERION OF n-DIMENSIONALITY AND AN APPROACH TO THE PROOF OF THE EQUALITY $\dim=\operatorname{Ind}$ FOR METRIC SPACES

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Abstract

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A CRITERION OF n-DIMENSIONALITY AND AN APPROACH TO THE PROOF OF THE EQUALITY $\dim = \operatorname{Ind}$ FOR METRIC SPACES

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§ 1. Formulation, preliminary information, explanations. Criterion.

For a T_1-space X to be homeomorphic to a metric space whose dimension dim* does not exceed n, it is necessary and sufficient that X possess a regular base, the boundaries** of whose elements form a family of multiplicity*** not greater than n.

Having the concept of a regular base and the metrization criterion from ([3]), this assertion can be reduced to known (though very deep) facts of the general theory of dimension.

Our main purpose is to show that the formulation given above is not merely curious, but also has a number of significant advantages. The transparent general technique of arguments with regular bases (see ([3])), applied in the present situation, makes it possible to make them a convenient connecting link in the proof of the fundamental Katětov-Morita theorem ([5,6]) on the equality of the dimensions dim and Ind of metric spaces. Morita's proof of this theorem ([5]) (see also ([1])) relies essentially on difficult special sum theorems for the dimension Ind of metric spaces, which, after the equality dim = Ind has been proved, become consequences of general theorems on the dimension dim, valid in the class of all normal spaces. In the argument given below, a minimum of properties of the dimension Ind is used; only those theorems on the dimensions dim and Ind that concern arbitrary normal spaces are widely applied, together with one assertion about paracompacts. More precisely, we assume the following facts to be known:

A. Let X be a paracompact space and $\dim X \leq n$. If $\{U_\alpha : \alpha \in \Gamma_i\}$, $i = 1, 2, ...$, are locally finite families of open subsets of X, and $\{F_\alpha : \alpha \in \Gamma_i\}$, $i = 1, 2, ...$, are families of closed sets, with $F_\alpha \subset U_\alpha$ for all $\alpha \in \bigcup\{\Gamma_i : i = 1, 2, ...\}$, then there exist open sets V_α such that: 1) $F_\alpha \subset V_\alpha \subset [\overline{V_\alpha}] \subset U_\alpha$; 2) $\mathrm{Ord}\{\mathrm{Fr}\, V_\alpha : \alpha \in \bigcup\{\Gamma_i : i = 1, 2, ...\}\} \leq n$.

This fundamental assertion is proved very transparently in the book ([1]), Chap. II, § 5, B) and C), pp. 23-27.

B. Let X be a normal space and let $\{F_\alpha : \alpha \in M\}$ be a locally finite family of closed subsets of X. Then, if $\dim F_\alpha \leq n$ for all $\alpha \in M$, also

$$\dim \bigcup\{F_\alpha : \alpha \in M\} \leq n.$$

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* By dim we denote dimension in the sense of coverings, or Lebesgue dimension. For the definitions of dim, ind, and Ind, see ([1],[2]).

** The multiplicity of a family γ of subsets of a point x is also denoted below by $\mathrm{Ord}_x\, \gamma$ and is defined as the cardinality of the set of all elements of γ containing x. The symbol n will everywhere below denote some natural number, 0 or -1. The notation $\mathrm{Ord}\, \gamma \leq n$ is equivalent to $\mathrm{Ord}_x\, \gamma \leq n$ for all $x \in X$.

*** The boundary $[U] \setminus U$ of the set U will be denoted below by $\mathrm{Fr}\, U$.

Transparent and direct proofs of this and of the following assertions can be found in ([2]), Ch. 7 (Theorems 1 and 2). Theorem 3, proved in ([2]), Ch. 7, simultaneously with Theorem 2, is the only serious fact on which Nagata relies in his proof of assertion A.

B. If X is a normal space and $X = \bigcup\{F_i : \ i = 1, 2, ...\}$, where $\dim F_i \leq n$ for all i and all F_i are closed subsets of X, then $\dim X \leq n$.

C. For any normal space X, $\dim X \leq \mathrm{Ind}\, X$.

This is a theorem of N. B. Vedenisov (see ([2]), Ch. 7, § 1, Theorems 11 and 12, and § 2, Theorem 4).

D. If F is a closed subspace of a normal space X, then $\mathrm{Ind}\, F \leq \mathrm{Ind}\, X$.

This elementary assertion follows obviously from the definition of Ind.

Remark. I do not know whether assertion A is true for arbitrary normal spaces.

Let us now recall (see ([3])):

Definition. A base $\mathcal{B}$ of a topological space X is called **regular** if, for every point x and every neighborhood Ox of it (and every closed subset F of X not containing x), there exists a neighborhood $O'x$ of the point x such that the set of all elements of the base $\mathcal{B}$ meeting both $O'x$ and $X \setminus Ox$ (and F) is finite.

E. **Metrization criterion** ([3]). A T_1-space X is metrizable if and only if it has a regular base.

We shall need the following elementary assertions about regular bases (their proofs can be found in ([3]). They are easily proved directly or derived from P5).

P1. *If* $\{\gamma_i : \ i = 1, 2, ...\}$ *is a sequence of locally finite open coverings of a metric space* X*, and for each* i *the diameter of every element of the covering* γ_i *does not exceed* $1/i$*, then* $\mathcal{B} = \bigcup\{\gamma_i : \ i = 1, 2, ...\}$ *is a regular base of the space* X*.*

P2. *Every regular base is the union of a countable family of locally finite systems of sets (the converse is false).*

P3. *If* $\mathcal{B}$ *is a regular base of the space* X*, and* $\mathcal{B}'$ *is a base of the space* X*, with* $\mathcal{B}' \subset \mathcal{B}$*, then* $\mathcal{B}'$ *is a regular base of the space* X*.*

P4. *If* X' *is a subspace of the space* X *and* $\mathcal{B}$ *is a regular base of* X*, then* $\mathcal{B}' = \{U \cap X' : \ U \in \mathcal{B}\}$ *is a regular base of the space* X'*.*

P5. *If* $\mathcal{B}$ *is a regular base of the space* X*, then* $m\mathcal{B} = \{U \in \mathcal{B} : \ \text{from } V \in \mathcal{B} \text{ and } U \subset V \text{ it follows that } U = V\}$ *is a locally finite covering of the space* X*. (*$m\mathcal{B}$ *is the set of all maximal elements of the base* $\mathcal{B}$*.)*

P6. *Let* $\mathcal{B}$ *be a regular base of the space* X*,* γ *an open covering of it, and*

$$\mathcal{B}_\gamma = \{U \in \mathcal{B} : \ \text{there exists } V \in \gamma \text{ for which } U \subset V\}.$$

Then $m\mathcal{B}_\gamma$ *is a locally finite open covering of the space* X*, inscribed in* γ*. (Note that* $\mathcal{B}_\gamma$ *is a regular base of the space* X*; see P3.)*

P6′. *Whatever open covering* γ *of a space* X *having a regular base* $\mathcal{B}$*, there exists a locally finite covering* λ *of this space, inscribed in* γ*, all elements of which belong to* $\mathcal{B}$*.*

It is precisely P6′ that is the specific property of regular bases which allows us to introduce a certain order into the proof of the Katětov-Morita theorem and to obtain advantages.*

* Nagata characterized metric spaces in terms of σ-locally finite bases $\mathcal{B}$ for which $\operatorname{Ord}\{\operatorname{Fr} U : \ U \in \mathcal{B}\} \leq n$. Such bases are a much weaker formation than regular bases: every countable base is σ-locally finite (trivially), but not from every countable base of the space of irrational numbers can one extract a regular one (see ([3])). Moreover, even countable bases may fail to have property P6. It is precisely this circumstance that makes it impossible to use σ-locally finite bases in the proof of the Katětov-Morita theorem not mediated by nontrivial sum theorems for Ind.

§ 2. The main argument

VI. *If* X *is a metric space and* $\dim X \leq n$*, then there exists a regular base* $\mathcal{B}$ *of the space* X *for which*

$$\operatorname{Ord}\{\operatorname{Fr} U : U \in \mathcal{B}\} \leq n^*.$$

Proof. It is known that in X there exists a sequence $\{\gamma_i : i = 1, 2, ...\}$ of open locally finite covers of the space X such that, if $U \in \gamma_i$, then the diameter of the set U does not exceed $1/i$. Let $\gamma_i = \{U_\alpha : \alpha \in \Gamma_i\}$; we shall assume that $\Gamma_i \cap \Gamma_j = \Lambda$ for $i \neq j$. For every $\alpha \in \Gamma_i$ one can find a closed set F_α in X such that $F_\alpha \subset U_\alpha$ and $\bigcup\{F_\alpha : \alpha \in \Gamma_i\} = X$ (this is a well-known property of point-finite open covers of normal spaces). By A, there exists a family $\{V_\alpha : \alpha \in \bigcup\{\Gamma_i : i = 1, 2, ...\}\}$ of sets open in X such that

$$F_\alpha \subset V_\alpha \subset [V_\alpha] \subset U_\alpha$$

for every $\alpha \in \Gamma_i$, and

$$\mathrm{Ord}\{\mathrm{Fr}\, V_\alpha : \alpha \in \bigcup \Gamma_i\} \leq n.$$

But from the trivial equality

$$\{V_\alpha : \alpha \in \bigcup \Gamma_i\} = \bigcup\{V_\alpha : \alpha \in \Gamma_i;\ i = 1, 2, ...\}$$

and P1 it follows that $\{V_\alpha : \alpha \in \bigcup \Gamma_i\}$ is a regular base of the space X. Thus,

$$\mathcal{B} = \{V_\alpha : \alpha \in \bigcup \Gamma_i\}$$

is the required base.

VII. *If a T_1-space X has a regular base $\mathcal{B}$ for which*

$$\mathrm{Ord}\{\mathrm{Fr}\, U : U \in \mathcal{B}\} \leq n,$$

then $\mathrm{Ind}\, X \leq n$.

Proof. We shall take the statements "$\mathrm{Ind}\, X = -1$," "$\dim X = -1$," "X is the empty set," "X has a regular base $\mathcal{B}$ for which

$$\mathrm{Ord}\{\mathrm{Fr}\, U : U \in \mathcal{B}\} \leq -1$$

" as equivalent. We now prove VII by induction. For $n = -1$, VII is true by virtue of the convention adopted by us. Suppose that the assertion VII has been proved for all $n \leq k$, and prove it for the case $n = k + 1$.

Thus, let $\mathcal{B}$ be a regular base of the space X and

$$\mathrm{Ord}\{\mathrm{Fr}\, U : U \in \mathcal{B}\} = k + 1.$$

Put

$$\mathcal{B}(U) = \mathcal{B} \mid \mathrm{Fr}\, U = \{V' = U' \cap \mathrm{Fr}\, U : U' \in \mathcal{B}\}.$$

If

$$x \in \mathrm{Fr}\, V' = [U' \cap \mathrm{Fr}\, U] \setminus (U' \cap \mathrm{Fr}\, U),$$

then: a) $U' \cap \mathrm{Fr}\, U \neq \Lambda$ and, hence, $U' \neq U$; b) $x \in [U'] \setminus U'$, i.e. $x \in \mathrm{Fr}\, U'$.

Consequently,

$$\mathrm{Ord}\{\mathrm{Fr}\, V' : V' \in \mathcal{B}(U)\} \leq k,$$

and, by P4, $\mathcal{B}(U)$ is a regular base of the space $\operatorname{Fr} U$. Using the induction hypothesis, we conclude that $\operatorname{Ind} \operatorname{Fr} U \leq k$. By Γ, then

$$\dim \operatorname{Fr} U \leq k$$

for all $U \in \mathcal{B}$. Let now $F \subset G$, where F is closed and G is open in X. Choose, using the normality of X, an open set G' for which

$$F \subset G' \subset [G'] \subset G.$$

Consider the binary cover $\gamma = \{G',\, X \setminus F\}$ of the space X. By P6′, there exists a locally finite cover λ of the space X, composed of elements of the base $\mathcal{B}$ and inscribed in γ. Obviously, if $W \in \lambda$ and $W \cap F \neq \Lambda$, then $W \subset G'$ (this follows from the fact that λ is inscribed in γ). Put

$$\Gamma = \bigcup \{W \in \lambda : W \cap F \neq \Lambda\}.$$

Then $\Gamma \supset F$ (for λ covers F), $\Gamma \subset G'$ (see above), and

$$[\Gamma] \neq \bigcup \{[W] : W \in \lambda \text{ and } W \cap F \neq \Lambda\}$$

by the local finiteness of λ. Hence

$$[\Gamma] \subset [G'] \subset G$$

and

$$\operatorname{Fr} \Gamma = [\Gamma] \setminus \Gamma = \bigcup \{[W] : W \in \lambda \text{ and } W \cap F \neq \Lambda\} \setminus \bigcup \{W : W \in \lambda \text{ and } W \cap F \neq \Lambda\}$$

$$\subset \bigcup \{[W] \setminus W : W \in \lambda \text{ and } W \cap F \neq \Lambda\} \subset \bigcup \{\operatorname{Fr} W : W \in \lambda\}.$$

But $\{\operatorname{Fr} W : W \in \lambda\}$ is a locally finite family of closed sets, and

$$\dim \operatorname{Fr} W \leq k$$

for all $W \in \lambda$ (for $W \in \mathcal{B}$).

By B, then,

$$\dim \bigcup \{\operatorname{Fr} W : W \in \lambda\} \leq k.$$

It now follows from VI that in the space

$$X' = \bigcup \{\operatorname{Fr} W : W \in \lambda\}$$

there exists a regular base $\mathcal{B}'$ for which

$$\operatorname{Ord} \{\operatorname{Fr} V : V \in \mathcal{B}'\} \leq k.$$

Using (again!) the induction hypothesis, we conclude that

$$\operatorname{Ind} X' \leq k.$$

But $\operatorname{Fr}\Gamma$ is a closed subspace of the space X' (see above). Hence, by D,

$$\operatorname{Ind}\operatorname{Fr}\Gamma \leq k.$$

This proves VII.

Proof of the Morita-Katětov theorem. Let X be a metric space. Comparing VI and VII, we conclude that

$$\operatorname{Ind} X \leq \dim X.$$

Together with Γ this means that

$$\operatorname{Ind} X = \dim X.$$

Proof of the criterion. The assertion VII, in which, on the basis of the Katětov-Morita theorem proved above, $\operatorname{Ind} X$ is replaced by $\dim X$, the assertion VI, and the metrization criterion E together constitute our criterion. This completes the main argument.

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* If a family D of sets is such that

$$\operatorname{Ord}\{\operatorname{Fr} A : A \in D\} \leq n,$$

we shall sometimes say that the peripheral multiplicity of the family D does not exceed n.

§ 3. **Remarks.** If X is a regular space with a countable base, then $\dim X \leq n$ if and only if X has a base $\mathcal{B}$ for which $\operatorname{Ord}\{\operatorname{Fr} U : U \in \mathcal{B}\} \leq n$, i.e., in the formulation of the criterion one may omit the word "regular." *

After what has been said above, the reader will easily prove this assertion (one should carry out the argument in the language of ind and then use the equality dim = ind).

In the case of arbitrary metric spaces we can no longer discard the assumption of regularity of the base. The conditions "X has a base $\mathcal{B}$ for which $\operatorname{Ord}\{\operatorname{Fr} U : U \in \mathcal{B}\} \leq 0$" and "$\operatorname{ind} X = 0$" are identical in meaning, but we know (see (7)) that there exists a metric space X for which $\operatorname{ind} X = 0$ and $\dim X = 1$.

By induction the following assertion is easily proved:

VIII. *If a topological space X has a base $\mathcal{B}$ for which $\{\operatorname{Fr} U : U \in \mathcal{B}\} \leq n$, then* $\operatorname{ind} X \leq n$.

In connection with this, the following two problems arise.

Problem 1. Is the converse to assertion VIII true in the class of metric spaces?

Problem 2. Let X be a metric space and $\operatorname{ind} X = n$. Is it then true that $X = \bigcup\{X_i : i = 1, 2, \dots, n+1\}$, where $\operatorname{ind} X_i = 0$ for all $i = 1, \dots, n+1$?

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* This is a slightly modified formulation of an assertion due to Nagata ([1]).

Note: Figure translations are in progress. See original paper for figures.